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Generalized Isolated Points and Generalized Perfect Sets in Topological Spaces

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النقاط المعزولة المعممة والمجموعات التامة المعممة في الفضاءات التبولوجية

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Abstract
This paper introduces the concepts of generalized isolated points (g-isolated points) and generalized perfect sets (g-perfect sets) within the framework of topological spaces, using the notions of generalized open and generalized closed sets. In this context, the concept of generalized scattered spaces (g-scattered spaces) is also defined. A series of fundamental results is established to clarify various properties of these generalized topological notions. Furthermore, the study provides several characterizations and presents illustrative examples that reveal their structure and demonstrate their connections with certain classical topological notions.

Keywords: g-isolated point, g-closed set, g-open set, g-perfect set, g-scattered space.

الملخص
يقدم هذا البحث دراسة لمفاهيم النقاط المعزولة المعممة والمجموعات التامة المعممة في الفضاءات التبولوجية، وذلك اعتماداً على مفهومي المجموعات المفتوحة المعممة والمجموعات المغلقة المعممة. وفي هذا السياق، تم ايضاً تقديم مفهوم الفضاءات المتناثرة المعممة ودراسة خواصها. كما تم في هذا البحث إثبات مجموعة من النتائج الأساسية لتوضيح خصائص هذه المفاهيم التبولوجية المعممة. علاوة على ذلك، يقدم البحث عدة خواص وأمثلة توضيحية تبرز البنية الأساسية لهذه المفاهيم وتُبين علاقاتها ببعض المفاهيم التبولوجية الكلاسيكية.

الكلمات المفتاحية: نقطة معزولة معممة، مجموعة مفتوحة معممة، مجموعة مغلقة معممة، مجموعة تامة معممة، مجموعة متناثرة معممة.

Introduction

Generalized notions in topological spaces have played a central role in extending classical results and enriching the understanding of topological structures. Levine [1] was the first to introduce and investigate the concept of generalized closed sets (or g-closed sets). Since then, several topological concepts based on g-closed sets have been developed and extensively studied by many researchers (see, for example, [2-6]).

In a topological space (X, τ), a point $p \in S \subseteq X$ is called an isolated point of S if there exists an open set U such that $\{p\} = S \cap U$. A closed set in X without isolated points is called perfect. X is said to be scattered if every nonempty subset of X has at least one isolated point (see, for example, [7,8]). The author in [9] defined and studied semi-perfect sets and semi-isolated points in topology, based on the concept of semi-open sets defined in [10].

In this paper, we use the notions of g-closed and g-open sets to define new concepts, including g-isolated points, g-perfect sets, and g-scattered spaces. The notion of g-isolated points provide more flexible conditions than the classical one, thereby generalizing the notion of isolated points in topological spaces. Several interesting results and properties related to these generalized concepts are established and studied.

1. Preliminaries

In this section, author presents some preliminary definitions and results. Throughout this paper, X denotes a topological space.

Definition 1. [1] A set $A \subseteq X$ is called generalized closed (briefly, g-closed) if $\bar{A} \subseteq V$ whenever V is open and $A \subseteq V$. The complement of a g-closed set is called generalized open (briefly, g-open).

Author denotes by $\mathcal{GO}(X)$ and $\mathcal{GC}(X)$ the collections of g-open and g-closed subsets of X, respectively.

Observation 1. [1] (1) Each closed set is g-closed.

(2) The union (respectively, the intersection) of finitely many g-closed (respectively, g-open) sets is g-closed (respectively, g-open).

(3) The intersection (respectively, the union) of finitely many g-closed (respectively, g-open) sets need not be g-closed (respectively, g-open).

Definition 2. [1] A space (X, τ) is called a $T_{\frac{1}{2}}$ - space if $\mathcal{GO}(X) = \tau$.

Observation 2. [1] Every T_1-space is $T_{\frac{1}{2}}$, and every $T_{\frac{1}{2}}$-space is T_0.

Theorem 1. [11] For each $p \in X$, either $\{p\}$ is closed or $X \backslash \{p\} \in \mathcal{GC}(X)$.

Definition 3. [11] Let $A \subseteq X$. The set $\bar{A}^g = \cap \{F: F \in \mathcal{GC}(X) \text{ and } A \subseteq F\}$ is called the g-closure of A.

Definition 4. A function $h: X_1 \to X_2$ is called

(1) g-irresolute [2] if $h^{-1}(F)$ is g-closed in X_1 whenever F is g-closed in X_2. Equivalently, if $h^{-1}(G)$ is g-open in X_1 whenever G is g-open in X_2.

(2) gc-homeomorphism [12] if *h* is bijective and both *h* and h^{-1} are g-irresolute.

The next Lemma is likely well-known, similar to the theorem related to the usual closure. Author provides its proof for completeness (see, [1,11,13]).

Lemma 1. Let $A \subseteq X$, then $p \in \bar{A}^g$ if and only if $G \cap A \neq \phi$ for every g-open set G that contains p.

Proof. Suppose that $A \cap G = \phi$ for some g-open set G containing p, then $A \subseteq G^c$. It follows that $\bar{A}^g \subseteq G^c$ since G^c is g-closed. Thus, $p \notin \bar{A}^g$. Conversely, suppose that $p \notin \bar{A}^g$. Since $\bar{A}^g = \cap \{H: H \text{ is } g-\text{closed}, A \subseteq H\}$, there must be a g-closed set H^* which contains A and $p \notin H^*$. Hence, $A \cap H^{*c} = \emptyset$, where H^{*c} is a g-open set containing p.

2. Generalized Isolated Points in Topological Spaces

Definition 5. Let $A \subseteq X$, a point $p \in A$ is called a generalized isolated point of A (briefly, g-isolated) if there exists $G \in \mathcal{GO}(X)$ such that $A \cap G = \{p\}$.

Author denotes the set of all g-isolated points of A by $I_g(A)$.

Observation 3. All isolated points of a set are g-isolated but not conversely. For example, let $A \subseteq X$, where X is an infinite indiscrete space and $|A| > 1$. Then each point of A is g-isolated in A but not isolated.

Lemma 2. Let X be a topological space.

(1) If $\{A_i\}_{i=1}^n$ is a finite family of subsets of X, and if p is a g-isolated point of A_i for each i, then p is a g-isolated point of $\bigcup_{i=1}^n A_i$.

(2) If $\{A_\alpha\}_{\alpha \in I}$ is any family of subsets of X, and if p is a g-isolated point of $\bigcup_{\alpha \in I} A_\alpha$, then p is g-isolated in A_{α_0} for some $\alpha_0 \in I$.

(3) If $\{A_\alpha\}_{\alpha \in I}$ is any family of subsets of X, and if p is a g-isolated point of A_α for each $\alpha \in I$, then p is g-isolated in $\bigcap_{\alpha \in I} A_\alpha$.

Proof. (1) Let p be a g-isolated point of A_i for each $i \in \{1, \dots, n\}$. Then, for each i there exists $G_i \in \mathcal{GO}(X)$ such that $\{p\} = A_i \cap G_i$. If $G = \bigcap_{i=1}^n G_i$, then $G \in \mathcal{GO}(X)$ and $\{p\} = (\bigcup_{i=1}^n A_i) \cap G$, so p is a g-isolated point of $\bigcup_{i=1}^n A_i$, as required.

(2) Clearly, if $\{p\} = (\bigcup_{\alpha \in I} A_\alpha) \cap G$ for some $G \in \mathcal{GO}(X)$, then $\{p\} = A_{\alpha_0} \cap G$ for some $\alpha_0 \in I$.

(3) For each $\alpha \in I$, since p is a g-isolated point of A_α, there exists $G_\alpha \in \mathcal{GO}(X)$ such that $\{p\} = A_\alpha \cap G_\alpha$. Thus, $\{p\} = (\bigcap_{\alpha \in I} A_\alpha) \cap G^*$ for some $G^* \in \{G_\alpha\}_{\alpha \in I}$.

Observation 4. Lemma 2 (1) does not necessarily hold for an infinite collection of subsets of X, as illustrated in the next example.

Example 1. Let us consider the collection $\left\{\left[0,1-\frac{1}{m}\right] \cup \{1\}\right\}_{m=1}^{\infty}$ in the space $\mathbb{R}$. Then 1 is a g-isolated point of the set $[0,1-\frac{1}{m}] \cup \{1\}$ for each $m \in \mathbb{N}$. However, $\bigcup_{m=1}^{\infty} [0,1-\frac{1}{m}] \cup \{1\} = [0,1]$ and 1 is not a g-isolated point of $[0,1]$.

Observation 5. The example below illustrates that the converse of Lemma 2 (3) does not necessarily hold.

Example 2. Consider $X = \{2,4,5,6\}$ and $\tau = \{\phi, X, \{2,5,6\}, \{4,5,6\}, \{5,6\}\}$. If $A = \{2,4\}$, then 4 is a g-isolated point of A. But $A = \{2,4,5\} \cap \{2,4,6\}$ and 4 is not a g-isolated point of either $\{2,4,5\}$ or $\{2,4,6\}$.

Theorem 2. (1) Let $A_1, A_2 \subseteq X$, then

(i) $I_g(A_1) \cap I_g(A_2) \subseteq I_g(A_1 \cap A_2)$

(ii) $I_g(A_1 \cup A_2) \subseteq I_g(A_1) \cup I_g(A_2)$

(2) $I_g\left(I_g(A)\right) = I_g(A)$ for each $A \subseteq X$

Proof. The proofs of (i) and (ii) follow directly from Lemma 2, while the proof of (2) is straightforward.

Theorem 3. If $B \subseteq X$ has no g-isolated points, then $\bar{B}^g$ also has no g-isolated points.

Proof. Assume that $\bar{B}^g$ has a g-isolated point p, then $\{p\} = \bar{B}^g \cap G$ for some $G \in \mathcal{GO}(X)$. Since $p \in \bar{B}^g$ and G is g-open with $p \in G$, then from Lemma 1 author has $B \cap G \neq \phi$. Since $B \cap G \subseteq \bar{B}^g \cap G = \{p\}$, author has $B \cap G = \{p\}$, and hence B has a g-isolated point p.

From Definition 5, author can restate the definition of a g-isolated point of a space as follows.

Definition 6. A point $p \in X$ is g-isolated in X if $\{p\} \in \mathcal{GO}(X)$.

Observation 6. Clearly, if $p \in I_g(X)$ then $p \in I_g(A)$ for any $A \subseteq X$ containing p.

Example 3. Let $X = \{1,3,5,7\}$ and $\tau = \{\phi, X, \{1,5\}, \{3,7\}\}$, then $\mathcal{GO}(X) = \mathcal{P}(X)$, the power set of X. Therefore, every point in X is g-isolated.

Theorem 4. A point $p \in X$ is g-isolated in X if and only if either $\{p\}$ is open or $\{p\}$ is neither open nor closed in X.

Proof. Clearly, if $\{p\}$ is open then $p \in I_g(X)$. If $\{p\}$ is neither open nor closed, then by Theorem 1, $X\backslash\{p\} \in \mathcal{GC}(X)$, and hence $\{p\} \in \mathcal{GO}(X)$; so $p \in I_g(X)$. Conversely, if the condition above does not hold, then $\{p\}$ is closed and not open, which implies that $X\backslash\{p\}$ is open and not closed. In this case, the closure of $X\backslash\{p\}$ is X, which is not contained in the open set $X\backslash\{p\}$, and this means that $X\backslash\{p\} \notin \mathcal{GC}(X)$. Therefore, $\{p\} \notin \mathcal{GO}(X)$, and hence $p \notin I_g(X)$.

Corollary 1. $p \in I_g(X)$ if and only if either $\{p\}$ is not closed or $\{p\}$ is clepen in X.

Corollary 2. Let $\{X_i\}_{i=1}^{n}$ be a family of topological spaces. If b_i is a g-isolated point of X_i for each i, then $(b_1, \dots, b_n)$ is a g-isolated point of the product space $\prod_{i=1}^{n} X_i$.

Proof. Suppose that $(b_1, \dots, b_n)$ is not a g-isolated point in the space $\prod_{i=1}^{n} X_i$, then Theorem 4 implies that the set $\{(b_1, \dots, b_n)\}$ is closed and not open. Since $\{(b_1, \dots, b_n)\}$ is closed, author have:

$$\prod_{i=1}^{n}\{b_i\} = \{(b_1, \dots, b_n)\} = \overline{\{(b_1, \dots, b_n)\}} = \prod_{i=1}^{n}\overline{\{b_i\}},$$

and hence each $\{b_i\}$ is a closed subset of X_i. Since $\{(b_1, \dots, b_n)\}$ is not open, then there exists $j \in \{1, \dots, n\}$ such that $\{b_j\}$ is not open in X_j. Therefore, $\{b_j\}$ is closed and not open in X_j, so b_j is not g-isolated in X_j.

Observation 7. The converse of Corollary 2 does not hold in general; author illustrates this by the following example.

Example 4. Suppose (X_1, τ) and (X_2, τ') are topological spaces, where $X_1 = \{1, 2\}$, $\tau = \{\phi, X_1\}$, $X_2 = \{7,9\}$, and $\tau' = \{\phi, X_2, \{7\}\}$. The point $(1,9)$ is g-isolated in $X_1 \times X_2$, whereas 9 is not g-isolated in X_2.

Theorem 5. Suppose that $h: X_1 \to X_2$ is a continuous and open injection. If p is a g-isolated point in X_1, then its image $h(p)$ is g-isolated in X_2.

Proof. If $h(p)$ is not g-isolated in X_2, then from Theorem 4, $\{h(p)\}$ is closed and not open in X_2. Since h is continuous and open, $\{p\}$ is closed and not open in X_1, so p is not a g-isolated point in X_1.

Definition 7. A space X is called generalized scattered (briefly, g-scattered) if every nonempty subset of X has a g-isolated point.

Example 5. The indiscrete space is g-scattered, whereas the space $\mathbb{R}$ is not.

Observation 8. Every scattered space is g-scattered but not conversely, as shown by:

Example 6. Let $X = \{2,4,7,10\}$ and $\tau = \{\phi, X, \{2,10\}, \{4,7\}\}$. The space X is not scattered since $\{2,10\}$ has no isolated points. However, X is g-scattered since $\mathcal{GO}(X) = \mathcal{P}(X)$.

Theorem 6. If X is a finite topological space then X is g-scattered.

Proof. Let $\phi \neq A \subseteq X$. If $A \cap I_g(X) \neq \phi$, then there exists $p \in A \cap I_g(X)$, and by Observation 6, $p \in I_g(A)$. If $A \cap I_g(X) = \phi$, then, according to Theorem 4 and Observation 1, for each $q \in A$, $A\backslash\{q\}$ is a closed set in X. Choose $b \in A$ and define $H = (A\backslash\{b\})^c$, then $H \in \mathcal{GO}(X)$ and $A \cap H = \{b\}$, so $b \in I_g(A)$. Thus, in both cases, A has a g-isolated point.

Theorem 7. Suppose that h is a g-irresolute bijection from a space X_1 onto a g-scattered space X_2, then X_1 is also a g-scattered space.

Proof. Let $\phi \neq A \subseteq X_1$. Then $h(A)$ has a g-isolated point p, so $h(A) \cap G = \{p\}$ for some $G \in \mathcal{GO}(X_2)$, which implies that $A \cap h^{-1}(G) = \{h^{-1}(p)\}$. Since h is g-irresolute, then $h^{-1}(G) \in \mathcal{GO}(X_1)$. Hence, $h^{-1}(p)$ is a g-isolated point of A.

3. Generalized Perfect Sets

Author begins by providing the definition and properties of generalized perfect spaces, followed by a study of perfect sets.

Definition 8. A space X is called generalized perfect (briefly, g-perfect) if it has no g-isolated points; that is, $I_g(X) = \phi$.

Example 7. All infinite cofinite spaces are g-perfect.

Observation 9. If X is g-perfect then X is perfect, but not conversely. For example, any indiscrete space X with $|X| > 1$ is perfect but not g-perfect.

Theorem 8. A space X is g-perfect if and only if every singleton $\{p\}$ in X is closed and not open.

Proof. This follows immediately from Theorem 4.

Corollary 3. Every g-perfect space is both a T_1- space and a $T_{\frac{1}{2}}$-space.

Observation 10. Clearly, the converse of the above corollary is generally false; for example, any infinite discrete space is a T_1- space but not g-perfect.

Theorem 9. Any nonempty g-perfect space is infinite.

Proof. This result follows directly from Theorem 6.

It was shown in [7] that the cofinite topology on an infinite set X is the smallest topology that makes X a T_1-space, known as the minimal T_1 topology. Therefore, from Corollary 3 and Example 7, the following result follows.

Theorem 10. Suppose τ is the cofinite topology on an infinite set X. If $\tau' \subset \tau$, then (X, τ') is not a g-perfect space. That is, τ is the minimal topology for which X is g-perfect.

Theorem 11. Let $E \subseteq X$ be an open set. If X is a g-perfect space then the subspace E is g-perfect.

Proof. Suppose, for a contradiction, that E is not a g-perfect space, then E has a g-isolated point p. By Corollary 3, X is a T_1-space, which implies that E is also a T_1-space. Thus, from Theorem 4, $\{p\}$ is clopen in E. Since E is open, the singleton $\{p\}$ must also be open in X, which contradicts the assumption that X is g-perfect.

Observation 11. If $E \subseteq X$ is closed, then the subspace E need not be g-perfect. For example, if $E \neq \phi$ is a finite subset of an infinite cofinite space X, then E is closed in X and is discrete as a subspace. Therefore, E is not a g-perfect subspace.

Definition 9. A set $E \subseteq X$ is called generalized perfect (briefly, g-perfect) if it is g-closed and $I_g(E) = \phi$.

Example 8. Let $X = \mathbb{R} \cup \{s\}$, where $s \notin \mathbb{R}$, and define a topology τ on X by:

$$\tau = \{U \subseteq X : U = \phi \text{ or } U = V \cup \{s\}, V \text{ is open in } \mathbb{R}\}.$$

Then $\mathbb{R}$ is a g-perfect set in X.

Observation 12. According to Theorem 3, a space is g-scattered if it contains no nonempty g-perfect subsets.

Theorem 12. If $\{A_i\}_{i=1}^{n}$ is a finite family of g-perfect sets in X, then $\bigcup_{i=1}^{n} A_i$ is also g-perfect.

Proof. It follows directly from Observation 1 and Lemma 2 (2).

Observation 13. According to Observation 1 (3), for a family $\{A_i\}_{i=1}^{n}$ of g-perfect sets in X, $\bigcap_{i=1}^{n} A_i$ is not necessarily g-perfect.

Theorem 13. If $B \subseteq X$ is a g-perfect set and $E \subseteq B$ is clepen in X, then E is also a g-perfect set.

Proof. Suppose, for a contradiction, that E is not a g-perfect set, so E has a g-isolated point p, which means that there exists $G \in \mathcal{GO}(X)$ with $E \cap G = \{p\}$. Since E is open, then $\{p\} \in \mathcal{GO}(X)$. By Observation 6, it follows that $p \in I_g(B)$, which contradicts the assumption that B is g-perfect.

Theorem 14. Let $E \subseteq X$ be clepen in X. Then X is a g-perfect space if and only if both E and E^c are g-perfect sets.

Proof. If X is g-perfect, the proof is a direct consequence of Theorem 13. Conversely, if X is not g-perfect, then X contains a g-open set $\{p\}$. Since either E or E^c contains p, it follows that one of them is not g-perfect.

The proof of the next result is clear and thus not included here.

Theorem 15. Suppose $h: X_1 \to X_2$ is a gc-homeomorphism. If $E \subseteq X_1$ is a g-perfect set, then $f(E) \subseteq X_2$ is also a g-perfect set.

Conclusion

In this research, author has introduced and investigated new concepts in topological spaces based on Levine's concept of g-closed sets. The newly defined concepts of g-isolated points and g-perfect sets provide a new framework for studying various topological structures. Our investigation has shown the relationships between these generalized concepts and their corresponding classical topological ones, providing a richer understanding of topological spaces.

Additionally, these concepts can be further developed using other classes of open sets, such as semi-g-open, regular g-open, and regular semi-open sets, which offer possible directions for establishing new generalized structures.

Moreover, these generalized notions allow for possible extensions into other domains within topology such as bitopological spaces and supra topological spaces. These extensions could lead to further theoretical developments. Overall, this work provides the foundation for a more generalized framework for studying structural properties in topological spaces.

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