Afro-Asian Journal of Scientific Research (AAJSR)
المجلة الأفروآسيوية للبحث العلمي
E-ISSN: 2959-6505
Volume 3, Issue 4, 2025
Page No: 31-40
Website: https://aajsr.com/index.php/aajsr/index
SJIFactor 2024: 5.028 ISI 2025: 0.915 معامل التأثير العربي (AIF) 2025: 0.76

Rethinking Negative Multiplication: Separating Magnitude from Direction

Omar Ali Zargelin*
Office of Innovation and Entrepreneurship Projects, Libyan Authority for Scientific Research, Tripoli, Libya

إعادة النظر في الضرب السالب: فصل المقدار عن الاتجاه

عمر علي زرغلين*
مكتب مشاريع الابتكار وريادة الأعمال، الهيئة الليبية للبحث العلمي، طرابلس، ليبيا

*Corresponding author: zargelin@aonsrt.ly

Received: August 02, 2025	Accepted: October 26, 2025	Published: November 01, 2025

Abstract:
Arithmetic rests on four fundamental operations, yet multiplication with negatives poses interpretational challenges that addition and subtraction do not. Historically, negative multiplication was introduced to preserve algebraic consistency and closure, but its tangible, real-world mechanism remains opaque: identities such as $(-a) \times b = -(a \times b)$and $(-a) \times (-b) = ab$ are algebraically valid while lacking an intuitive model that explains how the final sign arises as a realized operation. Building on prior work that re-read division by zero and division by negatives treating some actions as nonexistent rather than merely undefined this paper re-examines the legitimacy of negative multiplication as a real operation rather than a symbolic convention. We adopt a sign–magnitude split during computation with post-hoc sign restoration and propose a two-model framework. Model I bans direction × direction: squaring a negative preserve its sign, even roots of negatives are not realized on $\mathbb{R}$, and negative × negative is not a realized operation. Model II allows computation with direction preservation when two negatives meet, defining even roots of negatives as the negative of the positive root on the magnitude. The framework cleanly separates outcomes that are realized on $\mathbb{R}$from those that are symbolic only (requiring $\mathbb{C}$or falling outside the domain), without altering classical algebraic truths. The result is a principled boundary between symbolic structure and physical meaning, a transparent labeling policy for instruction, and a coherent pathway to complex analysis when and only when its introduction is conceptually warranted.

Keywords: Negative Multiplication, Sign–Magnitude Split, Realized Vs Symbolic Only, Even Roots Of Negatives.

الملخص
ترتكز الحسابات على أربع عمليات أساسية، غير أنّ الضرب مع السوالب يطرح تحديات تفسيرية لا تظهر في الجمع والطرح. تاريخيًا، أُدخل الضرب بالسالب لحفظ الاتساق الجبري وإغلاق المنظومة، لكن آليته الواقعية الملموسة ظلّت غير واضحة: فهويات مثل $(-a) \times b = -(a \times b)$ و $(-a) \times (-b) = ab$ صحيحة جبريًا لكنها تفتقر إلى نموذج حدسي يشرح كيف تنشأ الإشارة النهائية كعملية متحقّقة. انطلاقًا من عملٍ سابق أعاد قراءة القسمة على الصفر والقسمة بالسالب مع اعتبار بعض الأفعال غير موجودة لا مجرد غير معرّفة—تعيد هذه الورقة فحص مشروعية الضرب بالسالب كعملية واقعية لا كاصطلاح رمزي فحسب. نتبنّى أثناء الحساب تفكيك الإشارة / المقدار مع استعادةٍ للإشارة بعديًا، ونقترح

إطارًا ثنائيّ النماذج .النموذج الأوّل يحظر اتجاه × اتجاه: تربيع السالب يحافظ على إشارته، الجذور الزوجية للسوالب غير متحقّقة على ℝ، وسالب × سالب ليس عملية متحقّقة .النموذج الثاني يَسمح بالحساب مع حفظ الاتجاه عند التقاء سالبين، ويعرّف الجذور الزوجية للسوالب بأنها سالب الجذر على المقدار. يفصل هذا الإطار بوضوح بين النتائج المتحقّقة على ℝوتلك الرمزية فقط المحتاجة إلى ℂأو الواقعة خارج النطاق، من دون المساس بالحقائق الجبرية الكلاسيكية. والنتيجة حدودٌ منضبطة بين البنية الرمزية والمعنى الفيزيائي، وسياسةُ وسمٍ تعليمية شفافة، ومسارٌ متماسك نحو التحليل المركّب عندما وفقط عندما تقتضي الضرورة المفاهيمية ذلك.

الكلمات المفتاحية: الضرب بالسالب، تفكيك الإشارة / المقدار، متحقّق مقابل رمزي فقط، الجذور الزوجية للأعداد السالبة.

Introduction

This paper builds directly on our prior study, Rethinking Negative Multiplication: Separating Magnitude from Direction (Paper 1). There we proposed treating negativity primarily as direction rather than as a merely signed magnitude and showed how classroom confusion often arises from sliding sometimes unconsciously between purely symbolic rules and real-world operations. Here we extend that framework to three focal points: the status of negative × negative, powers of negative numbers (including non-integer exponents), and even roots of negative numbers. Our aim is not to alter algebraic validity but to clarify when a result is operationally realized on the real line and when it is symbolic-only, so instruction and applications can transparently mark the boundary.

Our guiding convention is a sign-magnitude split during computation, followed by controlled sign restoration. Concretely, we compute on magnitudes in the nonnegative domain while tracking direction separately; only at the end do we restore the sign according to explicit rules tied to the scenario (kinematic direction, financial inflow/outflow, charge, etc.). For instance, the standard distributive understanding of a negative factor across a positive factor remains intact at the level of magnitude, with the sign handled independently:

$$a \times (-b) \;=\; -(a \times b) \qquad (1)$$

To make the convention precise, we adopt a reusable decomposition for any $x \in \mathbb{R}$into direction and magnitude:

$$x \;=\; \operatorname{sgn}(x)\ |x|, \operatorname{sgn}(x) \in \{-1, +1\}, \quad |x| \geq 0 \qquad (2)$$

Equations (1) and (2) will be referenced throughout without repetition. Building on Paper 1, we now formalize a rulebook that distinguishes (i) operations whose realization on negative inputs is well-grounded in real settings from (ii) operations that are algebraically permissible yet lack a coherent real mechanism when both operands encode direction i.e., “direction × direction” treated as a realized operation rather than a symbolic composition. Statements relying on results proved in Paper 1 will be cited as “Paper 1, Sec. 11,” and only new or refined arguments are proved here.

Pedagogically, this yields two immediate benefits. First, learners carry out calculations on magnitudes simple, concrete steps while deferring the directional decision to an explicit restoration rule; this reduces cognitive load and avoids conflating algebraic syntax with physical meaning. Second, instructors and practitioners gain a principled way to label certain outputs symbolic only whenever the underlying operation does not correspond to a realizable process in the intended real domain (e.g., even roots of negatives on ℝ).

The remainder proceeds as follows. We first state the operational principles and a precise realized vs. symbolic-only criterion derived from the sign–magnitude convention and context-based restoration. We then analyze three case families negative × negative, negative bases under non-integer exponents, and even roots of negatives showing where the model supports realization and where it calls for symbolic-only tagging. A comparative section contrasts this reading with the classical symbolic approach, followed by implications for instruction and modeling. We close with open questions and a roadmap for extending the framework to exponential/logarithmic forms and other composite operations.

Background

This paper builds directly on our prior study, *Rethinking Negative Multiplication: Separating Magnitude from Direction* (Paper 1, under review, 2025). Paper 1 distinguished algebraic validity from the real-world realization of an operation on the real line, showing that much classroom confusion arises when symbolic rules are over-read as distributive/operational mechanisms once negativity as direction is involved. Here we briefly review the algebraic facts that matter for negatives, indicate where real interpretations become nontrivial, and explain how a sign-magnitude split complements rather than replaces the standard account, preparing a two-model framework that shares the split but makes different operational choices.

Field-axiomatically, $(-a) \times (-b) = ab$follows from distributivity and order compatibility; since $(-1) \times (-1) = +1$, we obtain $(-a) \times (-b) = ab$for all real a, b. Pedagogically, learners often reinterpret this as if "a negative time a negative creates positivity," whereas positivity here is a structural consequence of the axioms, not a physical mechanism flipping direction. Paper 1 argued that tracking direction (sign) separately from magnitude, computing on magnitudes, and restoring the sign by an explicit, context-tied rule reduces such misreadings.

For exponents, standard identities are already delimited when real values exist. For reduced rationales $p/q$with $p, q \in \mathbb{Z}_{>0}$,

$$x^{p/q} = \sqrt[q]{x^p} \tag{3}$$

so, a negative base yields a real value only when $q$is odd:

$$(-a)^{p/q} \in \mathbb{R} \text{ iff } q \text{ is odd, } a > 0 \tag{4}$$

and even roots summarize the same restriction:

$$\sqrt[2k]{-a} \notin \mathbb{R}, a > 0, k \in \mathbb{Z}_{>0} \tag{5}$$

In real only teaching and modeling, expressions falling under (5) or under (4) with even $q$are typically shunted to $\mathbb{C}$. Paper 1 recommended explicitly tagging such outcomes symbolic only unless complex-analytic reading is intended, while keeping all algebraic identities intact.

In sum, the present paper refines rather than replaces the classical account: we retain (3) -(5) but add a clear use-regime distinguishing realized from symbolic only operations in direction-laden scenarios. Building on the same split, we later introduce two models: Model I bans direction × direction, thereby not realizing negative × negative or even roots of negatives on $\mathbb{R}$; Model II allows computation with direction preservation when two negatives meet and defines even roots of negatives as the negative of the positive root on the magnitude, while constraints (3) - (5) for exponents and roots remain in force.

The Problem in the Classical Model

The classical reading becomes problematic at the point where a symbolic rule is silently promoted to a real-world operation. Once negativity is treated as direction, pushing sign rules through intermediate steps as if they were realized operations breeds confusion: squaring is read as an automatic sign flip, even roots of negatives are sent straight to $\mathbb{C}$, and non-integer powers on negative bases yield outputs with no presence on $\mathbb{R}$. This gap between algebraic structure and realistic meaning is the focus of this section, paving the way for a clear operational policy in the sections that follow.

Symbolic Tension

Classical algebra asserts that squaring a negative yield a positive:

$$(-1)^2 = +1 \tag{6}$$

Algebraically this is correct; under a direction-based reading it is better seen as a magnitude-only operation (squaring 1), with direction × direction not treated as a realized step. Equation (6) therefore records a logical consequence of the algebraic system rather than a physical mechanism that flips direction (cf. the sign–magnitude split in (2) and the symbolic-only stance in [1]). For even roots of negatives, the expression is routinely transferred to $\mathbb{C}$instead of being declared non-real on $\mathbb{R}$:

$$\sqrt{-4} = 2i \tag{7}$$

This is correct in complex analysis but, on a real-only track, represents a domain extension rather than a realized operation; hence it is tagged symbolic only unless a complex interpretation is explicitly intended (see the real-valued restriction in (5) and discussion in [5], [6]). For non-integer exponents on negative bases, complex outputs arise whenever the reduced fractional exponent has an even denominator; for instance, $1.9 = 19/10$:

$$(-5)^{1.9} \approx 26.69 - 9.60i \tag{8}$$

Equation (4) already warns that $(-a)^{p/q}$is real only when $q$is odd. Thus (8) is no algebraic surprise; pedagogically, in real-only contexts it should be tagged symbolic-only unless branch choices in $\mathbb{C}$are explicitly adopted ([1], [5], [6]).

Philosophical Tension

Negativity encodes direction, not an independent magnitude. Using the standard decomposition

$$x = \operatorname{sgn}(x) \mid x \mid (2)$$

We compute on $|x|$ and restore direction by explicit rules tied to context. Reading (6) as a self-caused direction change over-interprets what algebra supplies. In a realized, direction-aware interpretation, direction × direction is not an actual operation; statements like (6) and the general negative × negative = positive are algebraic consequences of field axioms, not causal mechanisms generating positivity. We therefore separate algebraic truth from the realizability of direction-laden operations ([1]). Raising a magnitude with direction should not, on a real only reading, alter direction unless a sign-restoration rule warrants it. Accordingly, we compute the power on the magnitude and then consult the restoration rule: if the case falls under the symbolic-only regime on $\mathbb{R}$(as in (7) and (8)), no new real direction is asserted; the operation is not realized unless we move explicitly to complex analysis (cf. (3) – (5) and [2 – 6]). This framing prepares the two-model treatment used later: in Model I, sign compounding is banned and even roots of negatives are not realized on $\mathbb{R}$; in Model II, sign restoration may preserve negative direction where specified, while even-denominator or irrational exponents on negative bases remain out of the real domain.

Principles of the Two Model Framework

The overarching idea is to separate **magnitude** from **direction** during computation and to forbid sign composition as an internal step. We always compute on a positive magnitude and postpone the sign decision to the end via an explicit policy rule. Within this separation, we introduce two distinct models: **Model I**, which **bans negative × negative** and, as a consequence, does **not** define the square root of a negative number on $\mathbb{R}$; and **Model II**, which **allows** computation while **preserving the negative direction** when two negatives meet, thereby providing a definition of the square root of a negative number consistent with that policy.

Separating Magnitude from Direction

Every negative input is represented as a positive magnitude with a fixed negative direction during the computation, and the sign is restored only at the end:

$$a_{\text{signed}} = -|a| \tag{9}$$

The prohibition principle states that multiplying/compounding signs is not a valid internal operation; signs are assigned and restored, not algebraically combined mid-process. This ensures semantic transparency: numerical work happens on magnitudes only, followed by a sign decision consistent with the chosen policy.

Model I: Ban on Negative × Negative

In Model I, $(-a) \times (-b)$is **disallowed on** $\mathbb{R}$because it instantiates direction × direction, which the prohibition principle forbids. To remain semantically coherent, familiar operations are specialized as follows:

- **Squaring a negative** is performed on the magnitude and the input's direction is restored at the end:

$$x^2 = \operatorname{sgn}(x)\ |x|^2 \tag{11a}$$

Hence $(-2)^2 = -4$. Squaring is not always positive here; it preserves the input's direction at the restoration step.

- **Even roots of negative numbers** are undefined on $\mathbb{R}$in this model, since an even root would invert a squaring step that retained direction. For example, $\sqrt{-9}$is undefined on $\mathbb{R}$.
- **Non-integer powers with a negative base**: $(-a)^r$is undefined on $\mathbb{R}$for $r \notin \mathbb{Z}$, $a > 0$, consistent with the prohibition principle.
- **Negative × Negative**: $(-a) \times (-b)$is undefined on $\mathbb{R}$with $a, b > 0$. No positive sign is generated by compounding two negatives.

Because squaring preserves direction under (11a), there is no direction-reversing even root within the same domain; consequently, the square root of a negative number is not available in Model I.

Model II: Allowed with Negative-Direction Preservation

Model II keeps magnitude-direction separation but makes a different policy decision when two negatives meet: the negative direction is preserved. We multiply magnitudes and then restore a negative sign:

- **Squaring a negative** follows the same operational template:

$$x^2 = \operatorname{sgn}(x)\ |x|^2 \tag{11b}$$

Thus $(-2)^2 = -4$. Squaring again preserves direction.

- **Even roots of negative** numbers are defined to be the negative of the root on the magnitude:

$$\sqrt{-a} = -\sqrt{a}(a > 0) \tag{12b}$$

So $\sqrt{-9} = -3$. This is not a return to the classical view; it is a definition chosen to be consistent with direction preservation in Model II.

- **Non-integer powers with a negative base** remain undefined on $\mathbb{R}$for non-integer exponents, aligning with the domain policy.

$$(-\boldsymbol{a})^{r} \text{ undefined on } \mathbb{R} \text{ for } r \in \mathbb{R} \setminus \mathbb{Z},\, a > 0 \qquad (13)$$

- **Negative × Negative** is executed on magnitudes with a restored negative sign:

$$(-\mathrm{a}) \times (-\mathrm{b}) = -\,\mathrm{ab} \qquad (14)$$

Thus $(-3) \times (-4) = -12$. The pedagogical aim is to show that multiplication is not a walk on the number line producing a double flip of direction; rather, it is a magnitude operation whose direction is determined by the policy.

Table (1): Core examples under (11) - (14): Classical vs. Model I vs. Model II.

Operation	Classical	Model I (ban: no −×−; no even roots on $\mathbb{R}$)	Model II (allowed; preserve negative)
$(-2)^2$	$+4$	$-4\leftarrow$ via (11)	$-4\leftarrow$ via (11)
$\sqrt{-9}$	$3i$	not realized on $\mathbb{R} \leftarrow$ via (12), (27a)	$-3\leftarrow$ via (12)/(27b)
$(-5)^{1.9}$	complex	not realized on $\mathbb{R} \leftarrow$ via (13), (18)	not realized on $\mathbb{R} \leftarrow$ via (13), (18)
$(-3) \times (-4)$	$+12$	not realized $\leftarrow$ via (14), (23a)	$-12\leftarrow$ via (14), (23b)

Note: We adopt the sign–magnitude split $x = \mathrm{sgn}(x)\ |x|$(2); computations run on magnitudes, and the sign is restored once according to the model's policy.

Table 1 condenses the paper's core message into a side-by-side view. It shows how the same operation receives different labels and outputs under the three readings: Classical, Model I (ban: no direction × direction, no even roots of negatives on $\mathbb{R}$), and Model II (allowed with direction preservation). Because all computations run on magnitudes and the sign is restored once at the end, the table cleanly separates what is realized on $\mathbb{R}$ from what is symbolic-only / not realized. Pedagogically, it gives instructors a quick rubric for grading and discussion, and it helps students see why results diverge without changing the underlying algebraic identities.

Realistic Interpretation of Roots and Powers

The behavior of roots and non-integer powers depends on the chosen model. Under Model I (ban on negative × negative), squaring preserves the input's direction and even roots of negatives are not realized on $\mathbb{R}$. Under Model II (allowed with direction preservation), we compute on magnitudes and restore a negative sign when two negatives meet, which yields a consistent definition of even roots of negatives.

Even Roots

Model II (allowed with direction preservation):

Using the magnitude first, restored sign template (11b), we define:

$$\sqrt{-a} \;=\; -\,\sqrt{a}, a > 0 \qquad (15)$$

This enforces **two-way inverse consistency on all reals**:

$$\sqrt[2]{x^2} = x \text{ for all } x \in \mathbb{R} \qquad (16)$$

and generalizes to higher even roots:

$$\sqrt[2k]{-a} \;=\; -\,\sqrt[2k]{a}, a > 0,\, k \in \mathbb{Z}_{>0} \qquad (17)$$

Equations (15) – (17) keep computation on magnitudes and restore the negative direction, making even-root operations on negatives realized within this model without defaulting to $\mathbb{C}$.

Model I (ban):

Since squaring in (11a) preserves the input's direction, no even root on $\mathbb{R}$can invert that step for negative inputs. Thus:

- $\sqrt{-a}$ is not realized on $\mathbb{R}$, and (16) is read domain-restricted: $\sqrt[2]{x^2} = x$ when $x \geq 0$, and not realized when $x < 0$.
- Equation (17) does not apply to negatives within this model; even roots of negatives remain out of domain.

Fractional or Non-Integer Powers

In both models we distinguish:

- If $r = p/q$in lowest terms with odd q, the value on a negative base is realized on $\mathbb{R}$via the odd root on magnitudes combined with integer powers (with sign restoration consistent with each model). If $q$is even, the operation is not realized on $\mathbb{R}$:

$$(-a)^{p/q} \text{ is realized on } \mathbb{R} \iff q \text{ is odd; otherwise not realized.} \qquad (18)$$

If $r \in \mathbb{R} \setminus \mathbb{Z}$is irrational (or cannot be reduced to $p/q$with odd q), powers of a negative base are not realized on $\mathbb{R}$:

$$(-a)^r \text{ not realized on } \mathbb{R}, r \text{ irrational.} \tag{19}$$

Illustrative example:

$$(-5)^{1.9} = \text{not realized on } \mathbb{R} \tag{20}$$

Since $1.9 = 19/10$has an even denominator, placing it under the second branch of (18).
Consistency notes: In Model II the negative direction is restored after even-root/magnitude computations, so (15) - (17) apply. In Model I this restoration is disallowed for even roots, leaving squaring direction-preserving while no even-root inverse exists for negatives within $\mathbb{R}$.

Partial Model

The partial model restricts core arithmetic to nonnegative magnitudes only and defers all sign handling to a post-computation restoration step. Multiplication and exponentiation are executed on magnitudes; signs are restored at the end according to the chosen policy (ban vs. allowed with preservation).

Sign magnitude representation (shared):

$$x = \operatorname{sgn}(x) \mid x \mid, \operatorname{sgn}(x) \in \{+1, -1\}, \mid x \mid \geq 0 \tag{21}$$

Magnitude only multiplication (allowed core):

$$m_1 \odot m_2 = m_1 m_2, m_1, m_2 \geq 0 \tag{22}$$

Post multiplication sign restoration (model specific):

- **Model I (ban on direction × direction):**

$$\operatorname{Restore}(s_1, s_2; m) = \begin{cases} +m & \text{if } s_1 = s_2 = +1 \\ -m & \text{if exactly one of } s_1, s_2 \text{ equals } -1 \\ \text{undef.} & \text{if } s_1 = s_2 = -1 \end{cases} \tag{23a}$$

- **Model II (allowed with negative-direction preservation):**

$$\operatorname{Restore}(s_1, s_2; m) = \begin{cases} +m & \text{if } s_1 = s_2 = +1 \\ -m & \text{if exactly one of } s_1, s_2 \text{ equals } -1 \\ -m & \text{if } s_1 = s_2 = -1 \end{cases} \tag{23b}$$

Integer powers (unified magnitude-first definition):

Compute m^non magnitudes only, then **restore the original input's sign** (no internal sign-multiplication; no parity-based flipping):

$$(s, m)^n \quad (s, m^n) \overset{\text{restore}}{\rightarrow} s\, m^n, n \in \mathbb{Z} \tag{24}$$

Thus for $x < 0$, $x^n = \operatorname{sgn}(x) \mid x \mid^n$ for all integers n, consistent with $(11a - 11b)$.

Fractional / non-integer powers (the same policy as in Section 5):

If $r = \frac{p}{q}$ in lowest terms with odd q, the value on a negative base is realized on $\mathbb{R}$via the odd root on magnitudes plus integer powers (then restore the sign). If $q$is even, or $r$is irrational, the expression on a negative base is undefined on $\mathbb{R}$:

$$(-a)^{p/q} \text{ defined only when } q \text{ is odd; otherwise undefined.} \tag{25}$$
$$(-a)^r \text{ undefined on } \mathbb{R} \text{ if } r \notin \mathbb{Z} \text{ and cannot be reduced to } p/q \text{ with odd } q. \tag{26}$$

Even root restriction (model dependent):

- **Model I (ban):**

$$\sqrt[2k]{-a} \text{ undefined on } \mathbb{R}, a > 0, \mathbb{Z}_{>0} \tag{27a}$$

- **Model II (allowed with preservation):**

$$\sqrt[2k]{-a} = -\sqrt[2k]{a}, a > 0, k \in \mathbb{Z}_{>0} \tag{27b}$$

Table (2): Quick examples consistent with $(22) - (27)$.

Operation	Magnitude only computation	Model I (ban)	Model II (allowed, direction-preserving)	Reference
$+3 \times -4$	$3 \odot 4 = 12$	-12	-12	Single negative restored: (22), (23a)/(23b)
-3×-4	$3 \odot 4 = 12$	Undefined	-12	Direction × direction banned in Model I (23a); preserved negative in Model II (23b)
$(-2)^2$	$2^2 = 4$	-4	-4	Restore input sign after integer power: (24), consistent with (11)
$(-5)^{1.9}$	—	Undefined	Undefined	$1.9 = \frac{19}{10}$has even denominator ⇒ undefined by (25)
$\sqrt{-9}$	—	Undefined	-3	Even-root restriction: (27a) in I; $\sqrt{-a} = -\sqrt{a}$in II (27b)

Table 2 illustrate the magnitude-first computation with model-specific sign restoration. For $(+3) \times (-4)$, we compute $3 \odot 4 = 12$and restore a single negative to get -12, which is allowed in both models. For $(-3) \times (-4)$, Model I deems the operation undefined by (23a), whereas Model II restores a negative direction after multiplying magnitudes, yielding -12(by (23b)). For $(-2)^2$, we compute $2^2 = 4$ and then restore the input's sign to obtain -4 in both models, consistent with (11). For $(-5)^{1.9}$, since $1.9 = 19/10$ has an even denominator, the expression is undefined in both models (by (25)). For $\sqrt{-9}$, Model I labels it undefined by (27a), while Model II realizes it as $-\mathbf{3}$ via the even-root convention with sign restoration (by (27b)).

Comparison between the Classical Model and the Two Realistic Variants

We now contrast three readings: the Classical Model, Model I (ban on negative × negative; no even roots of negatives on $\mathbb{R}$), and Model II (allowed with direction preservation; even roots of negatives defined as negative of the positive root). All models compute on magnitudes, but only the realistic variants forbid internal sign-multiplication and restore the sign once at the end. The table cites the governing rules for each entry.

Core Comparison Table

Table (3): Classical vs. Model I vs. Model II core outputs.

Operation	Classical Model	Model I (ban: no −×−; no even roots on $\mathbb{R}$)	Model II (allowed with direction preservation)
$(-3)^2$	$+9$	not realized on $\mathbb{R}$ ← negative × negative blocked	-9 ← via $x^2 = \operatorname{sgn}(x) \mid x \mid^2$ (11)
$\sqrt{-16}$	$4i$	not realized on $\mathbb{R}$ ← even root blocked (15)/(27a)	-4← $\sqrt{-a} = -\sqrt{a}$(15)/(27b)
$(-2)^{0.5}$	imaginary	not realized on $\mathbb{R}$ ← denominator even (18)	not realized on $\mathbb{R}$ ← denominator even (18)
$(-4) \times (-5)$	$+20$	not realized ← direction × direction banned (23a)	-20← preserve negative direction (23b)

Reminder: (sign magnitude split): $x = \operatorname{sgn}(x) \mid x \mid$ (2)

As indicated in Table 3. Classical vs. Model I vs. Model II core outputs under the sign magnitude policy. This table aligns symbolic outputs with the magnitude first policy: squaring preserves input sign (11), even roots of negatives are out of domain in Model I but defined as negative in Model II (15), fractional exponents with even denominators (or irrational) on negative bases are not realized on $\mathbb{R}$ (18), (19), and negative × negative is banned in Model I but yields a negative in Model II (23).

Step-by-Step Explanations

A) Squaring a negative, $(-3)^2$:

- Classical: $(-3) \times (-3) = +9$.
- Realistic: compute on magnitude, then restore sign via (11):
-

$$(-3)^2 = \mathrm{sgn}(-3)\ |-3|^2 = -9 \tag{28}$$

Reading: no internal negative × negative; the sign is restored once at the end.

B) Even root of a negative, $\sqrt{-16}$:

- Classical: move to $\mathbb{C}$, get $4i$.
- Realistic: magnitude-first, then sign restoration (15):

$$\sqrt{-16} = -\sqrt{16} = -4 \tag{29}$$

Inverse consistency with (11): $\mathrm{Root}_2(\mathrm{Sq}(-4)) = -4$ (16).

C) Fractional exponent with even denominator, $(-2)^{0.5}$:

- Classical: imaginary $= i\sqrt{2}$.
- Realistic: **not realized** on $\mathbb{R}$since $0.5 = \frac{1}{2}$has even denominator (18):

$$(-2)^{0.5} \text{ not realized on } \mathbb{R} \tag{30}$$

D) Negative × negative, $(-4) \times (-5)$:

- Classical: $+20$.
- Realistic: “direction × direction” is out of domain per (23):

$$(-4) \times (-5) \text{ not realized in the realistic model} \tag{31}$$

7.3 Additional Examples Highlighting the Differences

A) Opposite-sign multiplication, $(+3) \times (-4)$:

- Classical: -12.
- Realistic: multiply magnitudes, then restore a single negative (22), (23):

$$3 \odot 4 = 12 \ \Rightarrow\ \mathrm{Restore}(+1, -1; 12) = -12 \tag{32}$$

B) Odd-denominator root/power, $(-27)^{1/3}$:

- Classical: -3.
- Realistic: allowed since the denominator is odd (18):

$$(-27)^{1/3} = -\sqrt[3]{27} = -3 \tag{33}$$

C) Higher even root, $\sqrt[4]{-81}$:

- Classical: imaginary.
- Realistic: not realized on $\mathbb{R}$(17):

$$\sqrt[4]{-81} \text{ not realized on } \mathbb{R} \tag{34}$$

D) Odd integer power of a negative, $(-2)^3$:

- Classical: -8.
- Realistic: raise the magnitude, then restore the base’s sign once (24):

$$|\mathbf{-2}|^3 = \mathbf{8} \ \Rightarrow\ \text{final direction negative} = -\mathbf{8} \tag{35}$$

Conceptual Summary

1. **Classical:** sign is intertwined with the arithmetic steps (hence “negative × negative = positive”).
2. **Realistic:** compute **only on magnitudes**, prohibit direction × direction, and restore sign once at the end using clear rules (11), (15), (23).

3. **Instructional payoff:** explicit labels **realized** vs **symbolic-only/not realized** on $\mathbb{R}$ especially for even roots of negatives and negative bases with even-denominator or non-integer exponents (18), (19).

Educational and Real-World Impact

On a real only track, early instruction can avoid imaginary numbers altogether while remaining faithful to algebra. In **Model I**, expressions that depend on even-denominator or non-integer exponents on negative bases are not realized on $\mathbb{R}$ per (18), (19), and even roots of negatives are not realized per (27a). In **Model II**, even roots of negatives are realized via a magnitude-first computation with negative sign restoration per (15) - (17), while even-denominator or non-integer exponents on negative bases remain not realized per (18), (19). This postpones $\mathbb{C}$to the stage where its purpose is conceptually clear.

The governing convention is simple: compute on magnitudes only, then restore the sign once at the end by context. Addition, magnitude-only multiplication (22), and powers on magnitudes (24) remain concrete, while the sign is applied post hoc using the restoration policy (23) together with the model-appropriate even-root convention (27a) in Model I, (27b) with (15) in Model II. This separation supports deeper understanding with less rote: students learn why an expression is realized, symbolic only, or out of domain, rather than memorizing ad-hoc exceptions. Cross-references to (11), (15), (18), (19), and (23) keep reasoning consistent.

The approach also aligns with physical modeling. Direction is neither multiplied nor squared; it is respected and restored from context motion, current, cash flow while computations describe magnitudes and the sign records orientation, mirroring real measurement pipelines. Assessment becomes clearer because solutions can be labeled immediately as Realized on $\mathbb{R}$, Symbolic-only / requires $\mathbb{C}$, or out of domain, making intended semantics explicit. When $\mathbb{C}$is introduced later, it appears as a chosen extension for specific tasks (e.g., periodic models, phasors), not a mysterious fix.

Classroom contrasts make the policy concrete. For $\sqrt{-16}$, the classical route yields $4i$; in Model I it is not realized (27a), while in Model II it is $-4$by (15). For $(-2)^{1/2}$, the classical label is “imaginary”; in both models it is not realized on $\mathbb{R}$ because the denominator is even (18). For $(-3)^2$, the classical result is $+9$; in both models the result is -9 via $x^2 = \mathrm{sgn}(x)\ |\,x\,|^2$ (11), avoiding negative × negative as an internal step.

Future Work

Future work aims to develop a full calculus and exponential framework within the sign–magnitude regime for both models, specifying domains on $\mathbb{R}$, behavior at zero, and chain/product rules with post-hoc sign restoration, together with criteria that separate realized identities from symbolic-only ones. This includes an explicit axiomatization: for Model I, a formal account of the ban (no − × −, no even roots of negatives on $\mathbb{R}$); for Model II, a precise description of direction-preserving multiplication, its instructional scope, and its non-distributivity with standard addition where applicable. On the empirical side, we propose controlled classroom studies across middle school, high school, and first-year university cohorts to compare error rates, time-to-solution, and transfer to word problems against a classical syllabus. Parallel software development will deliver interactive modules featuring a magnitude-only core engine, single-step sign-restoration visuals, and clear output labels Realized on $\mathbb{R}$, Symbolic-only, or out of domain with teacher dashboards and analytics. A follow-on paper will extend the account to negative exponents on negative bases, even roots under Model II’s convention, and a didactic reformulation that postpones i while mapping precisely when and why $\mathbb{C}$ is later introduced. Finally, we outline local consistency proofs for both models in simple logical settings, curate constrained counterexamples, and specify safe transition conditions between models, culminating in curricular pathways that adopt Model I alone or blend it in stages with Model II before a principled entry into complex analysis.

Conclusion

Classical instruction often asks learners to accept non-realistic outcomes for example, a negative direction becoming positive without a real mechanism, or the need to invoke a new number system to justify even roots of negatives. The two-model, sign magnitude framework restores arithmetic to a natural workflow: computations operate purely on magnitudes, direction is respected and restored at the end by a transparent rule, even roots recover real values via sign restoration rather than complex detours, and non-realized operations are left out of domain on $\mathbb{R}$. In Model I, negative × negative is banned and even roots of negatives are not realized on $\mathbb{R}$; in Model II, negative × negative is allowed with a negative result (direction preserved) and even roots of negatives are defined as the negative of the positive root on the magnitude. By making these boundaries explicit, the framework re-balances symbolic algebra with physical meaning and offers a practical tool for coherent teaching, clearer modeling, and a smoother on-ramp to complex analysis when and only when its purposes are conceptually warranted.

References

[1] *Rethinking Negative Multiplication: Separating Magnitude from Direction* (Paper 1), [Manuscript submitted for publication, Academic Journal of Science and Technology (AJST), Journal of Libyan Academy Janzour].
[2] Charles C. Pinter, *A Book of Abstract Algebra*, 2nd ed., Dover, 2010.
[3] National Council of Teachers of Mathematics (NCTM), *Principles and Standards for School Mathematics*, 2000.
[4] Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw–Hill, 1976.
[5] Paul J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$, 2nd ed., Princeton University Press, 2016.
[6] Tristan Needham, *Visual Complex Analysis*, Oxford University Press, 1998.