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A Mathematical Analysis of the Newton-Raphson Method for Solving Nonlinear Equations: Local Convergence Properties and the Impact of Initial Values

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تحليل رياضي لطريقة نيوتن-رافسون في حل المعادلات غير الخطية: الخصائص المحلية وتأثير القيم الابتدائية

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Abstract:
One of the most basic iterative methods for analyzing nonlinear equations is the Newton-Raphson method. With an emphasis on local convergence qualities and the impact of the initial guess, this paper provides a simplified mathematical analysis of the method when used to solve a single nonlinear equation. There is discussion of both theoretical and practical elements, including difficulties arising from delayed convergence or bad initial values. pointing out Newton-Raphson's speed and precision benefits as well as its real-world drawbacks. The findings show that the approach is useful for resolving nonlinear equations in mathematics and engineering applications when the proper convergence conditions are satisfied.

Keywords: Newton-Raphson Method, Nonlinear Equations, Numerical Analysis, Taylor Series, Numerical Convergence.

الملخص
تُعد طريقة نيوتن-رافسون واحدة من أبسط الطرق التكرارية المستخدمة في تحليل المعادلات غير الخطية. يركّز هذا البحث على خصائص التقارب المحلي وتأثير التخمين الابتدائي، حيث يقدّم تحليلاً رياضيًا مبسطًا للطريقة عند تطبيقها لحل معادلة غير خطية واحدة. يتناول البحث الجوانب النظرية والعملية، بما في ذلك الصعوبات التي تنشأ نتيجة بطء التقارب أو اختيار قيم ابتدائية غير مناسبة. كما يسلّط الضوء على مزايا طريقة نيوتن-رافسون من حيث السرعة والدقة، إلى جانب بعض القيود التي تحدّ من فاعليتها في التطبيقات العملية. وتشير النتائج إلى أن هذه الطريقة تُعد أداة فعالة لحل المعادلات غير الخطية في مجالي الرياضيات والهندسة، وذلك عند تحقق شروط التقارب المناسبة.

الكلمات المفتاحية: طريقة نيوتن-رافسون؛ المعادلات الغير خطية؛ التحليل العددي؛ متسلسلة تايلور؛ التقارب العددي.

Introduction:

Numerical methods are employed to approximate solutions of equations when exact analytical solutions are difficult or impossible to obtain. These methods work by constructing successive approximations that ideally converge to the actual solution of a given equation or system of equations [1,2]. Such approaches are particularly important in scientific and engineering applications, where equations often involve nonlinear or transcendental functions of the form $f(x) = 0$ in a single variable. For instance, many boundary value problems encountered in areas like kinetic theory of gases, elasticity, and other physical phenomena are eventually reduced to solving such equations [3].

Among the various root-finding techniques, one of the most prominent is Newton's method, also known as the Newton-Raphson method. Although Isaac Newton initially developed the method in the late 17th century, it was independently discovered and first published by Joseph Raphson in 1690 roughly two decades before Newton's own publication [4]. What sets this method apart from others is its reliance on both the function *f(x)* and its derivative *f′(x),* evaluated at arbitrary points [5]. Newton’s method is widely applicable and can be used to approximate the roots of both linear and nonlinear equations of any degree, making it one of the most powerful tools in numerical analysis [6].

The Nonlinear Equation:

Nonlinear equations are basic mathematical models that are applied in engineering, science, and a variety of natural phenomena. When there are nonlinear functions, such as trigonometric, exponential, or logarithmic functions, or when the terms in an equation have exponents other than one, the connection between the variables is said to be nonlinear [7].

Nonlinear equations are generally more difficult to solve than linear equations, as many of them do not have explicit analytical solutions [8]. This necessitates the use of numerical methods to obtain approximate solutions with acceptable accuracy. These methods include the Newton-Raphson method, the bisection method, the secant method, and other iterative techniques [9].

The Taylor Series:

An infinite series expansion of a function that is endlessly differentiable at a given point is known as a Taylor series. The function is expressed as the sum of its derivatives evaluated at that pointω , divided by factorial terms, and multiplied by matching powers of the difference between the variable and the point ω. The Taylor series of a function's general form is [10]:

$$f(x)=\frac{f(\omega)}{0!}+\frac{(x-\omega)f'(\omega)}{1!}+\frac{(x-\omega)^2 f''(\omega)}{2!}+\cdots+\frac{(x-\omega)^n f^n(\omega)}{n!} \qquad (1)$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(x-\omega)^n}{n!}f^n(\omega) \qquad (2)$$

Newton-Raphson method:

When the derivative of the function f is straightforward and easy to compute, the real roots of the equation $f(x)=0$ can be determined with high accuracy using the Newton-Raphson method. The fundamental concept of this method is attributed to the scientist Isaac Newton, while the modern formulation commonly used today is credited to Joseph Raphson[9].In order to derive the general formulation of the method, let it be denoted by r we assume that an initial approximation of the desired root is given by x_0, and that c represents the correction term that must be added to x_0 to obtain the root r [11]; that is

$$r=x_0+c \qquad (3)$$

$$f(x_0+c)=f(r)=0$$

Applying Taylor's expansion of the function f about x_0, I get:

$$f(x_0+c)=f(x_0)+cf'(x_0)+\frac{c^2}{2}f''(x_0+\theta c), \quad 0<\theta<1 \qquad (4)$$

Assuming that the value of (c) is sufficiently small, the term containing (c^2) can be neglected, leading to the following relation:

$$f(x_0)+cf'(x_0)=0 \qquad (5)$$

$$c_1=c=-\frac{f(x_0)}{f'(x_0)} \quad (6)$$

$$x_1=x_0+c_1=x_0-\frac{f(x_0)}{f'(x_0)} \qquad (7)$$

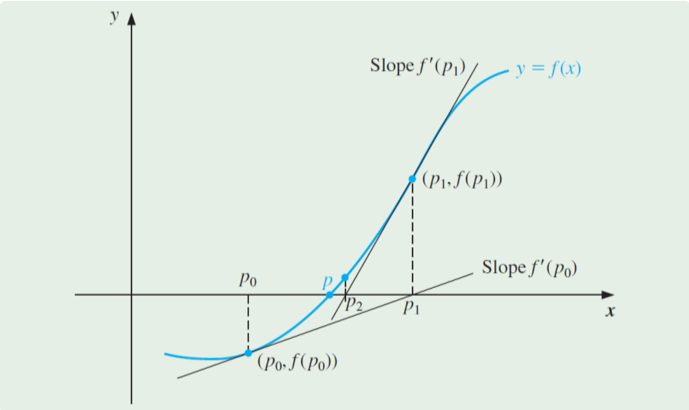

Figure (1): The operational mechanism of the Newton-Raphson method in approximating the roots of nonlinear equations.

An alternative way to derive the formula underlying Newton's method is by considering the slope of the tangent line at the point x_1, where this slope is equal to the derivative of the function at that point [12].

$$slope = f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad (8)$$

Suppose that $f(x_1) = 0$

$$f'(x_0) = \frac{-f(x_0)}{x_1 - x_0} \qquad (9)$$

Thus,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad (10)$$

Where $\frac{f(x_0)}{f'(x_0)}$ represents Δx the change in the value of x.
When this process is repeated, a sequence of points is obtained.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \ldots, x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (11)$$

Thus, the sequence approaches zero provided that $f'(x_n) \neq 0$.
Example:
Using the Newton-Raphson method to find a real root of the equation $f(x) = e^{-x} - \sin(\frac{\pi}{2}x)$, error value $\epsilon = 0.0001$.
Since $f(x) = e^{-x} - \sin(\frac{\pi}{2}x)$, the Newton-Raphson method says that approximate solutions should be generated by an iterative application $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$f'(x) = -e^{-x} - \frac{\pi}{2}\cos(\frac{\pi}{2}x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{e^{-x_n} - \sin(\frac{\pi}{2}x_n)}{-e^{-x_n} - \frac{\pi}{2}\cos(\frac{\pi}{2}x_n)}$$

I choose $x_0 = 1$, $Error = |x_{n+1} - x_n| < 0.0001$

$$x_1 = x_0 - \frac{e^{-x_0} - \sin(\frac{\pi}{2}x_0)}{-e^{-x_0} - \frac{\pi}{2}\cos(\frac{\pi}{2}x_0)} = 1 - \frac{e^{-1} - \sin(\frac{\pi}{2})}{-e^{-1} - \frac{\pi}{2}\cos(\frac{\pi}{2})} = -0.178$$

$$|x_1 - x_0| = |-0.178 - 1| = 1.178$$

$$x_2 = x_1 - \frac{e^{-x_1} - \sin(\frac{\pi}{2}x_1)}{-e^{-x_1} - \frac{\pi}{2}\cos(\frac{\pi}{2}x_1)} = -0.178 - \frac{e^{0.178} - \sin(\frac{\pi}{2}(-0.178))}{-e^{0.178} - \frac{\pi}{2}\cos(\frac{\pi}{2}(-0.178))}$$

$$x_2 = 0.665$$

$|x_2 - x_1| = |0.665 + 0.178| = 0.843$

$$x_3 = 0.665 - \frac{e^{-0.665} - \sin(\frac{\pi}{2}\ 0.665)}{-e^{-0.665} - \frac{\pi}{2}\cos(\frac{\pi}{2}\ 0.665)} = 0.4405$$

$|x_3 - x_2| = |0.4405 - 0.665| = 0.2245$

$$x_4 = 0.4405 - \frac{e^{-0.4405} - \sin(\frac{\pi}{2}\ 0.4405)}{-e^{-0.4405} - \frac{\pi}{2}\cos(\frac{\pi}{2}\ 0.4405)} = 0.4435$$

$|x_4 - x_3| = |0.4435 - 0.4405| = 0.003$

$$x_5 = 0.4435 - \frac{e^{-0.4435} - \sin(\frac{\pi}{2}\ 0.4435)}{-e^{-0.4435} - \frac{\pi}{2}\cos(\frac{\pi}{2}\ 0.4435)} = 0.4435$$

Real root of the equation $f(x) = e^{-x} - \sin\left(\frac{\pi}{2}x\right) = 0.4435$

Example:

Using the Newton-Raphson method to find a real root of the equation:

$f(x) = x^2 - 4\sin x$, with an error of $\epsilon = 0.0001$.

I choose $x_0 = 3$, $f'(x) = 2x - 4\cos x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$Error = |x_{n+1} - x_n| < \epsilon$$

$$x_{n+1} = x_n - \frac{x_n^2 - 4\sin x_n}{2x_n - 4\cos x_n}$$

$$x_1 = x_0 - \frac{x_0^2 - 4\sin x_0}{2x_0 - 4\cos x_0} = 3 - \frac{9 - 4sin3}{6 - 4cos3} = 2.1531$$

$|x_1 - x_0| = |2.1531 - 3|$ =0.8469

$$x_2 = x_1 - \frac{x_1^2 - 4\sin x_1}{2x_1 - 4\cos x_1} = 1.9540$$

$|x_2 - x_1| = |1.9540 - 2.1531|$ =0.1991

I continue working until I find the desired root.

Table (1): The results obtained after execution**.**

n	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$Error = \lvert x_{n+1} - x_n \rvert$
0	3.0000	2.1531	0.8469
1	2.1531	1.9540	0.1991
2	1.9540	1.9340	0.0200
3	1.9340	1.9338	0.0002
4	1.9338	1.9338	0.0000

Real root of the equation $f(x) = x^2 - 4\sin x = 1.9338$

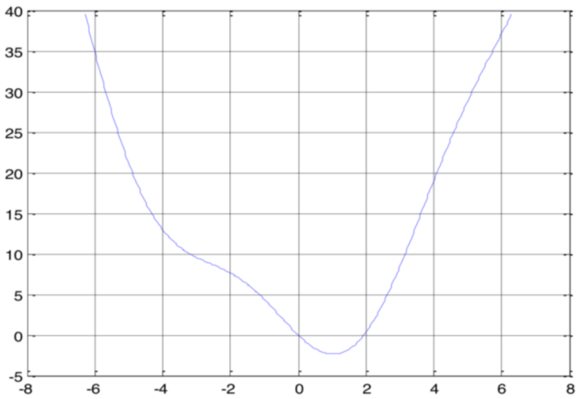

Figure (2): Function Graph $f(x) = x^2 - 4\sin x$

Notes on The Newton-Raphson Method:

When an appropriate initial estimate x_0 is selected near the actual root, the Newton-Raphson method converges rapidly and efficiently toward the solution. However, if the initial guess is not sufficiently close to the desired root, the method may fail to converge and instead diverge from the solution [13]. From the iterative formula used in the Newton-Raphson method, it can be observed that a larger value of the derivative $f'(x_0)$ results in a smaller correction term, thereby accelerating convergence toward the root. This implies that the method is particularly efficient when the tangent to the function at the pointx_0 is steep, i.e., nearly vertical [14].

Conversely, when $f'(x_0)$ is close to zero, the correction term becomes large, which can lead to slow convergence or complete divergence [15]. Consequently, the Newton-Raphson method is not recommended when the function's graph near the root is nearly parallel to the x-axis. In such cases, it is essential to select an initial approximation x_0 that lies very close to the exact root to enhance the likelihood of successful [16].

Conclusion:

The Newton-Raphson method represents an efficient and widely used numerical approach for locating the roots of a given function ($f(x)$), provided that the function is differentiable, and its derivative is explicitly known. The method aims to determine the value of the independent variable (x) that satisfies the condition ($f(x) = 0$). Beginning with an initial guess (x_n), the method employs the tangent to the function at that point to compute a refined estimate (x_{n+1}), determined by the point at which the tangent line intersects the x-axis. This iterative scheme is repeated until the root is approximated to a desired level of accuracy. Owing to its fast convergence properties and computational effectiveness, the Newton-Raphson technique is regarded as one of the most reliable and precise methods for solving nonlinear equations. Furthermore, Newton's method can be extended to handle systems of nonlinear equations by replacing the single derivative with the Jacobian matrix, allowing for the simultaneous solution of multiple equations in several variables.

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