Afro-Asian Journal of Scientific Research (AAJSR)
المجلة الأفروآسيوية للبحث العلمي
E-ISSN: 2959-6505
Volume 3, Issue 4, 2025
Page No: 145-149
Website: https://aajsr.com/index.php/aajsr/index
SJIFactor 2024: 5.028 ISI 2025: 0.915 معامل التأثير العربي (AIF) 2025: 0.76

# Geometric Properties of A New Subclass of Close-to-Convex Functions

Ebtisam A. Eljamal[1*], Huda Ali Aldweby[2], Zuhra. A. Abusutash[3], Ahlam H. Alrefai[4]
[1,3,4]Department of Mathematics, Faculty of Science, Al-Mergeb University, Al-Khums, Libya
[2]Department of Mathematics, Faculty of Science, Al-asmarya Islamic University, Zliten, Libya

# الخصائص الهندسية لفصل تكافؤ جديد للدوال القريبة التحدب

إبتسام علي الجمل[1*]، هدى علي الذويبي[2]، زهرة علي ابوسطاش[3]، أحلام حسن الرفاعي[4]
[1،3،4]قسم الرياضيات، كلية العلوم، جامعة المرقب، الخمس، ليبيا
[2]قسم الرياضيات، كلية العلوم، الجامعة الاسمرية، زليتن، ليبيا

*Corresponding author: eaeljamal@elmergib.edu.ly

| Received: September 08, 2025 | Accepted: November 27, 2025 | Published: December 08, 2025 |
| --- | --- | --- |



**Abstract:**
The authors explore the properties of analytic functions generated by a new integral operator in the open unit disk. The core of the work focuses on a particular differential inequality involving these functions. The main results demonstrate that satisfying this inequality forces the functions to have desirable geometric characteristics. Specifically, the paper derives practical conditions that ensure these functions are univalent (one-to-one), close-to-convex, and strongly starlike.

**Key Words:** Analytic function, Integral Operator, Close-To-Convex Function.

**الملخص:**
يتناول الباحثون خصائص الدوال التحليلية المُنتجة بواسطة مؤثر تكاملي جديد في القرص الوحدة المفتوح. يتمحور العمل حول دراسة متباينة تفاضلية محددة تتضمن هذه الدوال. تُظهر النتائج الرئيسية أن تحقيق هذه المتباينة يضمن امتلاك الدوال لخصائص هندسية مرغوبة. وتحديداً، تقدم الورقة شروطاً عملية تضمن أن تكون هذه الدوال أحادية (واحد لواحد)، وقريبة من التحدب، وشبيهة نجمية بشدة.

**الكلمات المفتاحية:** الدالة التحليلية، مؤثر تكاملي، الدوال شبه محدبة.

**Introduction and definitions:**
Let A denote the class of functions of the form:

$$\Theta(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_\kappa \xi^\kappa$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the conditions that $\Theta(0) = \Theta'(0) - 1 = 0$.

Close-to-convex functions represent a subclass of $A$. Their definition stipulates that for a function $f$ to be close-to-convex, there must exist a convex function $g$ and an angle $\alpha$ in $(-\pi/2, \pi/2)$ satisfying the condition $\Re(e^{i\alpha}\frac{\Theta'(\xi)}{g'(\xi)}) > 0$ at every point $z$ within the domain $U$.

It is a established result that all close-to-convex functions are univalent. Previous research [1, 2] demonstrated that an analytic function $f$ is not only close-to-convex but also univalent in $U$ if the real part of its derivative, $R(f'(z))$, is positive throughout the domain.
A function $f \in A$ qualifies as strongly starlike of order $\alpha$ $(0 \leq \alpha < 1)$ whenever the expression:

$$|\arg(\frac{\xi(z)}{\xi(z)})| < \frac{\alpha\pi}{2}.$$

This geometric condition is equivalent to the subordination relationship:

$$\frac{\xi\Theta'(\xi)}{\Theta(\xi)} \prec (\frac{1+\xi}{1-\xi})^{\alpha}$$

Given functions $\Theta, g \in A$, we introduced the the following integral operator where studied in (see [6] and [7]):

$$\mathcal{I}(\Theta, g)(\xi) = I_{\alpha}^{n}F(\xi) = \frac{\alpha+1}{\xi^{\alpha}}\int_{0}^{\xi}(\Theta(t)e^{g(t)})^{\alpha}\, dt \qquad (1)$$

Analytically, this operator is represented by the series, where $F(z) = (\Theta(z)e^{g(t)})$.

$$I_{\alpha}^{\kappa}F(\xi) = \xi + \sum_{\kappa=2}^{\infty}\frac{\alpha+1}{\alpha+\kappa}c_{\kappa}\xi^{\kappa}. \qquad (2)$$

The primary objective of this work is to establish criteria for the close-to-convexity and univalence of analytic functions generated by an integral operator. The same problem of analytic function was investigated by S.Billing in (see[5]). To prove our results, we need the following lemmas.
**Lemma 1.1** (see [4]) Let u and v denote complex variables:

$$u = u_1 + iu_2,\ v = v_1 + iv_2,$$

and let $(u, v)$ be a complex-valued function that satisfies the following conditions.
**(i):** $\psi(u, v)$ is continuous in domain $D$ of complex plane,
**(ii)**: $(0, 1) \in D$ and $\psi(1, 0) > 0$
**(iii):** $\Re\psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v, 1 \leq -\frac{1+u_2^2}{2}$.
Let $p(\xi) = 1 + p_1\xi + \ldots$ is an analytic function in $U$ such that $(p(\xi), \xi p'(\xi)) \in D$. If

$$\Re\{\psi(p(\xi), \xi p'(\xi))\} > 0$$

Then

$$\Re p(\xi) > 0 \in U.$$

**Lemma 1.2** (see[3]) Consider a convex function $G$ in the unit disk $U$, normalized by $G(0) = a$, and a complex number $\gamma$ where $\Re(\gamma) > 0$. If $F$ is a holomorphic function of the form $F(\xi) = a + a_n\xi^n + a_{n+1}\xi^{n+1} + \cdots$ in $U$ and $F$ is subordinate to $G$, then:

$$\frac{1}{\xi^{\gamma}}\int_{0}^{\xi}F(t)t^{\gamma-1}dt \prec \frac{1}{\kappa\xi^{\frac{\gamma}{n}}}\int_{0}^{\xi}G(t)t^{\frac{\gamma}{n}-1}dt.$$

**2 Main result:**
**Theorem 2.1** Consider real parameters $\alpha$ and $\beta$ such that $\alpha + 2\beta \geq 0$ and $\beta < 1$. Under these parameter constraints, the differential inequality:

$$\left|(1-\alpha)\frac{I_{\alpha}^{\kappa}F(\xi)}{\xi} + \alpha(I_{\alpha}^{\kappa}F(\xi))'\right| > \mu, \xi \in U \qquad (3)$$

implies the positivity condition $\Re(\frac{I_{\alpha}^{\kappa}F(\xi)}{\xi}) > 0$ for all $z$ in the unit disk.
**Proof**. Let $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \ldots$ be analytic function in U such that for all $\xi \in U$,

$$P(\xi) = \frac{I_{\alpha}^{\kappa}F(\xi)}{\xi}. \qquad (4)$$

Then

$$(1-\alpha)\frac{I_{\alpha}^{\kappa}F(\xi)}{\xi} + \alpha(I_{\alpha}^{\kappa}F(\xi))' = P(\xi) + \alpha\xi P'(\xi).$$

Equivalently, condition (3) can be written as:

$$\Re\left[\frac{p(\xi) + \alpha\xi p'(\xi) - \beta}{1-\beta}\right] > 0, \xi \in U \qquad (5)$$

If $D = \mathbb{C} \times \mathbb{C}$, define:

$$\psi(u, v) = \frac{1}{1-\beta}[u + \alpha v - \beta].$$

Then $(u, v)$ is continuous in $D$ and $\Re\psi(1,0) = 1 > 0$. Further, in view of (5), we get $\Re\psi(p(\xi), \xi p'(\xi)) > 0, z \in U$. let $u = u_1 + iu_2 = v_1 + iv_2$ where $u_1, u_2$, $v_1$ and $v_2$ are real numbers. Then for $iu_2$, $v_1 \in D$ with $v_1 \leq \frac{1+u_2^2}{2}$, we have:

$$\Re\psi(iu_2, v_1) = \Re\left[\frac{iu_2 + \alpha v_1 - \beta}{1-\beta}\right] = -\frac{\alpha + 2\beta}{2(1-\beta)} \leq 0.$$

Hence, by Lemma 1.1 and (4) we get $\Re P(\xi) > 0$ in $U$ or equivalent, $\Re\frac{I_\alpha^n F(\xi)}{\xi} > 0$.

**Theorem 2.2** Under the parameter constraints $\alpha + 2\beta \leq 0$ and $\beta > 1$, the differential inequality:

$$|(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi} + \alpha(I_\alpha^\kappa F(\xi))'| < \beta, \xi \in U, \qquad (6)$$

guarantees that $\frac{I_\alpha^\kappa F(\xi)}{\xi}$ has a positive real part throughout the unit disk.

**Proof**. Define the function $P(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots$ to be analytic in $U$, where:

$$P(\xi) = \frac{I_\alpha^\kappa F(\xi)}{\xi}. \qquad (7)$$

Differentiating this expression and substituting, we find:

$$(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi} + \alpha(I_\alpha^\kappa F(\xi))' = P(\xi) + \alpha\xi P'(\xi).$$

Consequently, the hypothesis that $I_\alpha^\kappa F(\xi)$ satisfies inequality (6) is equivalent to the condition:

$$|P(\xi) + \alpha\xi P'(\xi)| < \beta, \xi \in U.$$

This condition can be reformulated as:

$$\Re\left(\frac{P(\xi) + \alpha\xi P'(\xi) - \beta}{1-\beta}\right) > 0, \xi \in U. \qquad (8)$$

The remainder of the proof follows using the same methodology as in the proof of Theorem 2.1.

**Theorem 2.3** Given scalars α, β in **R** with the property that $\alpha > 1$ and $0 \leq \beta < 1$. For a function f obeying the bound.

$$\Re\left|(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi} + \alpha(I_\alpha^\kappa F(\xi))'\right| < \beta, \xi \in U$$

Implies that $\Re\frac{I_\alpha^\kappa F(\xi)}{\xi} > \frac{\beta}{\alpha}$. in U. Consequently, $I_\alpha^\kappa F(\xi)$ is a close-to-convex (and therefore univalent) function in the unit disk $U$.

**Proof.** On the basis of Theorem 2.1, for $0 \leq \beta < 1$ allows us to conclude that

$\Re\frac{I_\alpha^\kappa F(\xi)}{\xi} > 0$. Define

$$P(\xi) = (1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi} + \alpha\big(I_\alpha^\kappa F(\xi)\big)'.$$

Yields

$$\big(I_\alpha^\kappa F(\xi)\big)' = \frac{1}{\alpha}P(\xi) + \left(1 - \frac{1}{\alpha}\right)\frac{I_\alpha^\kappa F(\xi)}{\xi}$$

This establishes that:

$$\Re\frac{\big(I_\alpha^\kappa F(\xi)\big)'}{\xi} > \frac{\beta}{\alpha}.$$

**Theorem 2.4** Consider a convex function $H$ in the unit disk $U$, normalized such that $H(0) = 1$, and a real number $\alpha > 1$. If the integral operator $I_\alpha^\kappa F(\xi)$ fulfills:

$$(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi} + \alpha\big(I_\alpha^\kappa F(\xi)\big)' \prec H(\xi), \qquad \xi \in U, \quad (9)$$

yield

$$\frac{I_\alpha^\kappa F(\xi)}{\xi} \prec \frac{1}{\alpha\xi^{\frac{1}{\alpha}}}\int_0^\xi H(t)t^{\frac{1}{\alpha}-1}dt$$

Proof. Consider a regular function:

$$P(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots \text{ in } U$$

Defined by the relation:

$$P(\xi)=\frac{I_\alpha^\kappa F(\xi)}{\xi}. \qquad (7)$$

Differentiating and substituting, the condition (7) simplifies to the subordination:

$$P(\xi)+\alpha\xi P'(\xi)\prec H(\xi). \qquad (10)$$

Applying Lemma 1.2 with the parameter $\gamma=1/\alpha$, the subordination in (10) yields the solution:

$$P(\xi)\prec\frac{1}{\alpha\xi^{1/\alpha}}\int_0^\xi H(t)t^{1/\alpha-1}dt.$$

Consequently, by the definition of $P(z)$, we obtain the final result:

$$\frac{I_\alpha^\kappa F(\xi)}{\xi}\prec\frac{1}{\alpha\xi^{1/\alpha}}\int_0^\xi H(t)t^{1/\alpha-1}dt.$$

**Theorem 2.5** Consider $\alpha$ as a real number, with $\alpha>1$. For A function $I_\alpha^\kappa F(\xi)$ obeying the bound:

$$(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi}+\alpha\left(I_\alpha^\kappa F(\xi)\right)'\prec 1+\lambda\xi, \lambda>0, \qquad \xi\in U, \qquad (11)$$

Establishes that:

$$\frac{I_\alpha^\kappa F(\xi)}{\xi}\prec\frac{\lambda\xi}{\alpha+1}, \qquad (12)$$

As the same time:

$$\left(I_\alpha^\kappa F(\xi)\right)'\prec\frac{2\lambda\xi}{\alpha+1}.$$

**Proof.** Building on Theorem 2.4, the specific choice $H(\xi)=1+\lambda\xi$ leads directly to the conclusion (12) whenever the hypothesis (11) is satisfied.
Define the function:

$$(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi}+\alpha\left(I_\alpha^\kappa F(\xi)\right)'-1=\emptyset(\xi).$$

This definition allows us to express the derivative as:

$$\left(I_\alpha^\kappa F(\xi)\right)'-1=\frac{1}{\alpha}\emptyset(\xi)+\left(1-\frac{1}{\alpha}\right)\left(\frac{I_\alpha^\kappa F(\xi)}{\xi}-1\right).$$

By applying the bounds from (11), (12), and the equation above, we derive the inequality:

$$\left|\left(I_\alpha^\kappa F(\xi)\right)'-1\right|<\frac{\lambda\xi}{\alpha+1}.$$

This result immediately yields the following corollary.

**Corollary 2.1** Consider $\alpha$ and $\lambda$ parameters , with $\alpha>1$ and $0<\lambda\le\frac{\alpha+1}{2}$. For a given function $I_\alpha^\kappa F(\xi)$ obeying the bound:

$$\left|(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi}+\alpha\left(I_\alpha^\kappa F(\xi)\right)'-1\right|<\lambda, \ \xi\in U,$$

This yields $\left|\left(I_\alpha^\kappa F(\xi)\right)'-1\right|<\frac{2\lambda\xi}{\alpha+1}\le 1$. We conclude that $I_\alpha^\kappa F(\xi)$ is univalent in $U$, a direct consequence of its close-to-convex nature.

**Theorem 2.6** Consider $\alpha>1$. For a given function $I_\alpha^\kappa F(\xi)$ obeying the bound:

$$(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi}+\alpha\left(I_\alpha^\kappa F(\xi)\right)'\prec\frac{1-(1-\alpha)\xi}{(1-\xi)^2}, \qquad \xi\in U, \qquad (13)$$

It follows that:

$$\frac{I_\alpha^\kappa F(\xi)}{\xi}\prec\frac{1}{1-\xi} \qquad (14)$$

Additionally:

$$\Re I_\alpha^\kappa F(\xi)>\frac{3}{4}, \ \xi\in U.$$

**Proof.** In light of Theorem 2.4 for $H(\xi)=\frac{1-(1-\alpha)}{(1-\xi)^2}$ which is a convex function gives (14), whenever (13) holds. Let:

$$(1-\alpha)\frac{I_\alpha^\kappa F(\xi)}{\xi}+\alpha\left(I_\alpha^\kappa F(\xi)\right)'=Q(\xi).$$

Therefore,

$$\left(I_\alpha^\kappa F(\xi)\right)'=\frac{1}{\alpha}Q(\xi)+\left(1-\frac{1}{\alpha}\right)\left(\frac{I_\alpha^\kappa F(\xi)}{\xi}\right).$$

By using (13), (14) and the above equation, we obtain:

$$\Re I_{\alpha}^{\kappa} F(\xi) > \frac{3}{4}, \ \ \xi \in U.$$

**Conclusion:**

In this research the authors investigated some geometric properties of analytic functions generated by a new integral operator in the open unit disk and also the main result concentrated on particular differential in equality satisfied geometric characteristics specifically and practical conditions that ensure univalent (one- to – one), close-to- convex and strongly starlik functions.

**REFERENCES**


1. Noshiro, K., On the theory of schlicht functions, J. Fac. Sci., Hokkaido Univ.,**2**(1934-35), 129-155.
2. Warchawski, S. E., On the higher derivatives at the boundary in confor- mal mappings, Trans. Amer. Math. Soc*.,* **38**(1935), 310-340.
3. Hallenbeck, D. J. and Ruscheweyh, S., Subordination by convex func- tions, Proc. Amer. Math. *Soc.,* **52**(1975), 191-195.
4. Miller, S. S. and Mocanu, P. T., Differential subordinations and inequal- ities in the complex plane, J. Diff. Eqns., 67(1987), 199-211.
5. Sukhwinder, S. B., A subclass of close-to-convex functions, Int. J. Open Problems Complex Analysis*.,* **2**(2010), 2074-2827.
6. Maslina, D., Ebtisam, E. and Daniel, B., On a class of bounded starlike functions.
7. Maslina, D., Ebtisam, E. and Daniel, B., Some Results of New Univalent and Starlike Integral Operator.