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An Analytical and Applied Study on the Summation Properties of Σ-Strictly Diagonally Dominant Matrices with Numerical Verification

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دراسة تحليلية وتطبيقية لخصائص جمع المصفوفات المهيمنة قطريا بصرامة من النمط *Σ-SDD* مع تحقق عددي

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Abstract:
In this research, we extend the existing results on subdirect sums of S-strictly diagonally dominant matrices to the more general class of Σ-Strictly Diagonally Dominant (Σ-SDD) matrices. As Σ-SDD matrices form a proper superclass of S-SDD matrices and a subclass of H-matrices, we investigate their structural properties under matrix combination operations. Specifically, we provide a complete characterization of the conditions under which the k-subdirect sum of two Σ-SDD matrices preserves the Σ-SDD property. Furthermore, we establish when the conventional sum of two Σ-SDD matrices remains Σ-SDD. These results broaden the understanding of diagonal dominance under scaling and contribute to the study of matrix stability within hierarchical subclasses of H-matrices.

Keywords: k-subdirect sum, H-matrices, Σ-SDD matrices, overlapping blocks, diagonal dominance.

الملخص:
تهدف هذه الورقة إلى توسيع النتائج المتعلقة بالمجاميع الفرعية المباشرة للمصفوفات ذات الهيمنة القطرية الصارمة من النوع *S-SDD* لتشمل الفئة الأكثر عمومية، وهي المصفوفات ذات الهيمنة القطرية الصارمة من النوع *Σ-SDD*. وبوصف هذه الفئة تمثل فئة عليا (*superclass*) للمصفوفات *S-SDD* وفئة فرعية (*subclass*) من المصفوفات *H-matrices*، فإن الدراسة تبحث في البُنى الرياضية التي تحافظ على خاصية الهيمنة القطرية الصارمة تحت عمليات الجمع المختلفة. تقدم الورقة توصيفًا كاملاً للشروط التي يبقى عندها المجموع *k* الفرعي المباشر (-*k* *subdirect sum*) لمصفوفتين من النوع *Σ-SDD* محافظًا على خاصية *Σ-SDD*، كما تُبيّن الحالات التي يظل فيها المجموع التقليدي لمصفوفتين من النوع *Σ-SDD* محتفظًا بالخاصية نفسها. وتسهم هذه النتائج في تعزيز فهم استقرارية المصفوفات ضمن الفئات المتدرجة التابعة لـ *H-matrices*.

الكلمات المفتاحية: k-subdirect sum، *Σ-SDD*، المصفوفات *H-matrices*، المجاميع الفرعية المباشرة.

Introduction

The k-subdirect sum operation on square matrices arises naturally in several computational and algebraic contexts, where it provides a structured framework for combining matrix blocks while preserving key stability-related properties [2,8]. One of the most fundamental matrix classes associated with such operations is the class of strictly diagonally dominant (SDD) matrices [4], characterized by the requirement that each diagonal entry exceeds, in absolute value, the sum of all off-diagonal entries in the corresponding row.

Recent developments have extended this concept to the broader class of Σ-Strictly Diagonally Dominant (Σ-SDD) matrices [2], in which diagonal dominance is defined through a summation involving scaled or partitioned off-diagonal entries. The symbol "Σ" reflects the use of weighted or structured summations that generalize the classical SDD condition. As a result, Σ-SDD matrices form a proper superclass of S-SDD matrices and a subclass of H-matrices, the latter being characterized by comparison matrices that satisfy strict diagonal dominance [5,6].

Understanding how Σ-SDD matrices behave under matrix addition is crucial for applications in numerical analysis, iterative methods, and block-structured matrix computations. Specifically, determining the conditions under which the sum of two Σ-SDD matrices remains within the same class has important implications for matrix stability and convergence theory. Likewise, analyzing the preservation of the Σ-SDD property under the k-subdirect sum operation provides insight into how diagonal dominance behaves under structured block combinations.

Specifically, this research follows the theoretical framework of Bru and Cvetkovic, et al (2008)[2]. Our goal is to verify their findings on Σ-SDD matrix sums using a simplified notation and numerical examples, we build upon existing results for S-SDD matrices and extend them to the more general Σ-SDD framework. We begin by recalling the essential definitions and preliminary concepts needed throughout the paper [2,6,8].

Preliminaries and Definitions:

Definition 1: [6] A matrix $M = \left[m_{ij}\right] \in C^{n,n}$ is called an SDD matrix if, for each $i \in N$, it holds that

$$|m_{ii}| > r_i(M) \tag{1}$$

It is well known that.

Definition 2: Consider a square matrix $M = \left[m_{ij}\right] \in C^{n,n}$, where $n \geq 2$, Let Σ be a non-empty proper subset of the index set $N = \{1,2, \ldots .n\}$. We categorize M as a Σ-strictly diagonally dominant (Σ-SDD) matrix if the following inequalities are satisfied:

$$|m_{ii}| > r_i^{\Sigma}(M) \quad \text{for all } i \in \Sigma \text{ and}$$
$$(|m_{ii}| - r_i^{\Sigma}(M))|a_{jj}| - r_j^{\Sigma^c}(M)) > r_i^{\Sigma^c}(M) r_j^{\Sigma}(M) \quad \text{for all } i \in \Sigma, j \in \Sigma^c \tag{2}$$

If $\Sigma = M$ (and thus $\Sigma = \emptyset$), the conditions reduce to $|m_{ii}| > r_i(M)$ for all $i \in N$,. This is simply the standard definition of a strictly diagonally dominant (SDD) matrix. The definition above is a slight modification of the one given in [4], where the case $\Sigma = N$ was excluded. However, this omission is inconsequential: if a matrix is SDD, it automatically satisfies the Σ-SDD property for every nonempty proper subset $\Sigma \subset N$.

Definition 3: Let $M = \left(m_{ij}\right) \in C^{n \times n}$. If $|m_{ii}| > R_i(M), \forall_i \in N$, then M is said to be a strictly diagonally dominant matrix and denoted $M \in D$, If there exists a positive diagonal matrix X such that $MX \in D$, then M is said to be a generalized strictly diagonally dominant matrix (i.e. nonsingular H- matrix) and denoted $M \in D^*$.

Subdirect Sum of Matrices:

As in [4][8][2], Let M and N be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq min(n_1, n_2)$. Let M and N be partitioned into 2×2 blocks as follows,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{3}$$

where M_{11} and N_{22} are square matrices of order k. [8], we call the square matrix of order $n = n_1 + n_2 - k$ given by

$$P = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & M_{11} + N_{11} & N_{12} \\ O & N_{21} & N_{22} \end{bmatrix} \tag{4}$$

the k-subdirect sum of M and N and denote it by $P = M \oplus_k N$.

Each element p_{ij} of the resulting matrix $P = M \oplus_k N$ can be explicitly constructed from the elements of M and N. For precise formulation, we define the following index subsets

$$\alpha = \{1,2, \ldots, n_1 - k\},\ \beta = \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}, \gamma = n_1 + 1, n_1 - k + 2, \ldots, n \tag{5}$$

with note that $\alpha \cup \beta \cup \gamma = \{1,2, \ldots, n\}$ and that $n = t + n_2$.

Let $P = p_{(ij)}$ be a matrix, and by defining $t = n_1 - k$, we can write

$$p_{ij}=\begin{cases} m_{ij} & i\in\alpha, & j\in\alpha\cup\beta \\ 0 & i\in\beta, & j\in\gamma \\ m_{ij} & i\in\beta\,, & j\in\alpha \\ m_{ij}+n_{i-t,j-t} & i\in\beta, & j\in\beta \\ n_{i-t,j-t} & i\in\beta, & j\in\gamma \\ 0 & i\in\beta\,, & j\in\alpha \\ n_{i-t,j-t} & i\in\gamma\,, & j\in\beta\cup\gamma \end{cases} \quad (6)$$

Main Theorems for k-Subdirect Sums:

Theorem 1: [4] Consider two square matrices M and N with dimensions $n_1\times n_1$ and $n_2\times n_2$ respectively, where $n_1\geq 2$. let K be an arbitrary integer within the rang $1\leq K\leq min(n_1,n_2)$. Define the index sets $\alpha,\beta\,,\gamma$ as in (5). Partition M and N as in (3). Let Σ be a set of indices of the form $\Sigma=\{1,2,....\}$ suppose that:

1. M is Σ-strictly diagonally dominant, with $|\Sigma|\leq|\alpha|$.
2. N be strictly diagonally dominant.
3. All diagonal entries of M_{22} and N_{11} are positive (or all negative).

Then the k-subdirect sum $P=M\oplus_k N$ is Σ-strictly diagonally dominant, and consequently nonsingular.

Proof. We begin our prove the case which $\Sigma=\alpha$. Given the property that M is α-strictly diagonally dominant, it follows that:

For all $i\in\alpha$:

$$|m_{ii}|>r_i^\alpha(M)=\sum_{\substack{j\in\alpha\\ j\neq i}}|m_{ij}| \quad (7)$$

and for all $i\in\alpha, j\in\beta$:

$$\left(|m_{ii}|-r_i^\alpha(M)\right)\left(|m_{jj}|-r_j^\beta(M)\right)>r_j^\beta(M)r_j^\alpha(M) \quad (8)$$

Note that M is of order n_1; then the complement of α in $\{1,2,\ldots,n_1\}$ is β. We want to show that P is also an α-strictly diagonally dominant matrix, i.e., we must show that

1. $|p_{ii}|>r_i^\alpha(P)\forall i\in\alpha.$
2. $\left(|p_{ii}|-r_i^\alpha(P)\right)\left(|p_{jj}|-r_j^{\beta\cup\gamma}(P)\right)>r_i^{\beta\cup\gamma}(P)r_j^\alpha(P)\forall i\in\alpha,\forall j\in\beta\cup\gamma.$ (9)

Since P is of order $n=n_1+n_2-k$, the complement of α in $\{1,2,\ldots,n\}$ is $\beta\cup\gamma$.

Proof of condition (1): using equation (6), and part 1 of (9).

to obtain

$$|p_{ii}|=|m_{ii}|>r_i^\alpha(M)=r_i^\alpha(P),\forall i\in\alpha. \quad (10)$$

Proof of condition (2): We distinguish two cases: $j\in\beta$ and $j\in\gamma$.

If $j\in\beta$, (recall that $t=n_1-k$):

From (6) we have the following relations:

$$r_j^\alpha(P)=\sum_{j\neq k,k\in\alpha}|p_{jk}|=\sum_{j\neq k,k\in\alpha}|m_{jk}|=r_j^\alpha(M), \quad (11)$$

$$r_j^{\beta\cup\gamma}(P)=\sum_{j\neq k,k\in\beta\cup\gamma}|p_{jk}|=\sum_{j\neq k,k\in\beta}|p_{jk}|+\sum_{j\neq k,k\in\gamma}|p_{jk}|=r_j^\beta(P)+r_j^\gamma(P), \quad (12)$$

$$r_j^\beta(P)=\sum_{j\neq k,k\in\beta}|m_{jk}+n_{j-t,k-t}|, \quad (13)$$

$$r_j^\gamma(P)=\sum_{j\neq k,k\in\gamma}|n_{j-t,k-t}|=r_{j-t}^\gamma(N), \quad (14)$$

$$p_{jj}=m_{jj}+n_{j-t,j-t} \quad (15)$$

Therefore we can write

$$\left(|p_{ii}|-r_i^\alpha(P)\right)\left(|p_{jj}|-r_j^{\beta\cup\gamma}(P)\right)$$

$$=\left(|m_{ii}|-r_i^\alpha(M)\right)\left(|m_{jj}+n_{j-t,j-t}|-r_j^\beta(P)-r_j^\gamma(P)\right),\forall i\in\alpha,\forall j\in\beta \quad (16)$$

Starting with the substitution $p_{ii}=m_{ii}$, indices $i\in\alpha$ and equations (12) and (15). If the diagonal entries of both sub-matrices M_{22} and N_{11} carry the same sign whether positive or negative the absolute value of their sum simplifies directly to:

$$|m_{jj} + n_{j-t,j-t}| = |m_{jj}| + |n_{j-t,j-t}|$$

We can now update our main inequality. By substituting these new values (16) as:

$$\left(|p_{ii}| - r_i^{\alpha}(P)\right)\left(|p_{jj}| - r_j^{\beta\cup\gamma}(P)\right)$$
$$= \left(|m_{ii}| - r_i^{\alpha}(M)\right)\left(|m_{jj}| + |n_{j-t,j-t}| - r_j^{\beta}(P) - r_j^{\gamma}(P)\right), \forall i \in \alpha, j \in \beta \tag{17}$$

Next, we examine the right hand of the equation (17). By applying (13) the triangle inequality principle, we can establish an upper bound for the row sum as follows:

$$r_j^{\beta}(P) = \sum_{j\neq k, k\in\beta} |m_{jk} + n_{j-t,k-t}| \leq \sum_{j\neq k, k\in\beta} |m_{jk}| + \sum_{j\neq k, k\in\beta} |n_{j-t,k-t}|$$
$$= r_j^{\beta}(M) + r_{j-t}^{\beta}(N) \tag{18}$$

Using (14), from (18) we obtain the following inequality:

$$|m_{jj}| + |n_{j-t,j-t}| - r_j^{\beta}(P) - r_j^{\gamma}(P) \geq |m_{jj}| + |n_{j-t,j-t}| - r_j^{\beta}(M) - r_{j-t}^{\beta}(N) - r_{j-t}^{\gamma}(N), \tag{19}$$

Since we have $r_{j-t}^{\beta}(N) + r_{j-t}^{\gamma}(N) = r_{j-t}^{\beta\cup\gamma}$, we obtain

$$|m_{jj}| + |n_{j-t,j-t}| - r_j^{\beta}(P) - r_j^{\gamma}(P) \geq |m_{jj}| - r_j^{\beta}(M) + |n_{j-t,j-t}| - r_{j-t}^{\beta\cup\gamma}, \tag{20}$$

This allows us to transform (17) into:

$$\left(|p_{ii}| - r_i^{\alpha}(M)\right)\left(|m_{jj}| - r_j^{\beta\cup\gamma}(P)\right) \geq$$
$$\left(|m_{ii}| - r_i^{\alpha}(M)\right)\left(|m_{jj}| - r_j^{\beta}(M) + |n_{j-t,j-t}| - r_{j-t}^{\beta\cup\gamma}(N)\right), \forall i \in \alpha, \forall j \in \beta, \tag{21}$$

Since $\left(|m_{ii}| - r_i^{\alpha}(M)\right)$ is positive (as *M* is α-SDD) and $|n_{j-t,j-t}| - r_{j-t}^{\beta\cup\gamma}(N)$ is also positive (as N is SDD), we have,

$$|m_{jj}| - r_j^{\beta}(M) + |n_{j-t,j-t}| - r_{j-t}^{\beta\cup\gamma}(N) > |m_{jj}| - r_j^{\beta}(M) \tag{22}$$

This jointly with (18) leads to the strict inequality

$$\left(|p_{ii}| - r_i^{\alpha}(P)\right)\left(|p_{jj}| - r_j^{\beta\cup\gamma}(P)\right) > \left(|m_{ii}| - r_i^{\alpha}(M)\right)\left(|m_{jj}| - r_j^{\beta}(M)\right), \ \forall\, i \in \alpha, \forall\, j \in \beta \tag{23}$$

Finally, using 2 of (9) and equation (11):

$$\left(|m_{ii}| - r_i^{\alpha}(M)\right)\left(|m_{jj}| - r_j^{\beta}(M)\right) > r_i^{\beta}(M)\, r_j^{\alpha}(M) = r_i^{\beta\cup\gamma}(P) r_j^{\alpha}(P), \quad \forall\, i \in \alpha, \forall\, j \in \beta \tag{24}$$

which allows us to transform equation (23) into the inequality

$$\left(|p_{ii}| - r_i^{\alpha}(P)\right)\left(|p_{jj}| - r_j^{\beta\cup\gamma}(P)\right) > r_i^{\beta\cup\gamma}(P) r_j^{\alpha}(P), \forall i \in \alpha, \forall j \in \beta \tag{25}$$

If $j \in \gamma$,
From (6):

$$r_j^{\alpha}(P) = \textstyle\sum_{j\neq k, k\in\alpha} |P_{jk}| = 0. \tag{26}$$

Therefore, the condition 2 of (9) becomes

$$\left(|p_{ii}| - r_i^{\alpha}(p)\right)\left(|p_{jj}| - r_j^{\beta\cup\gamma}(P)\right) > 0, \quad \forall i \in \alpha, \forall j \in \gamma, \tag{27}$$

This inequality is fulfilled since:

The first term is positive: $|p_{ii}| - r_i^{\alpha}(P) = |m_{ii}| - r_i^{\alpha}(M) > 0.$

The second term of (27) is also positive: $p_{ii} = n_{j-t,j-t}$. for all $j \in \gamma$ and

$$r_j^{\beta\cup\gamma}(P) = \textstyle\sum_{j\neq k, k\in\beta\cup\gamma} |P_{jk}| = \sum_{j\neq k, k\in\beta\cup\gamma} |n_{j-t,k-t}| = r_{j-t}^{\beta\cup\gamma}(N), \forall j \in \gamma, \tag{28}$$

Since N is strictly diagonally dominant we have

$$|n_{j-t,j-t}| - r_{j-t}^{\beta\cup\gamma}(N) > 0, \forall j \in \gamma \tag{29}$$

Therefore equation (27) is fulfilled, which completes the proof for the case where $\Sigma = \alpha$. When $|\Sigma| < |\alpha|$, the proof is analogous. With the key point being the subcase $j \in \alpha\backslash\Sigma$, a condition similar to 2) for P in (9) still holds. We provide an illustrative example of theorem 1 with a practical application of the proof.

Example 1 Consider the following matrices:

$$M_1 = \begin{bmatrix} 8 & 1 & 0.5 \\ 0.5 & 7 & 1 \\ 0.5 & 0.5 & 6 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} 6 & 0.5 & 0.5 \\ 0.5 & 9 & 1 \\ 0.5 & 1 & 8 \end{bmatrix}, \tag{30}$$

with $n_1 = n_2 = 3,\ k = 1$, and $\Sigma = \alpha = \{1,2\}$. Matrix M_1 is α-SDD, matrix N_1 is SDD, and the diagonal entries of $(M_1)_{22}$ and $(N_1)_{11}$ are positive. The 1-subdirect sum

$$P_1 = M_1 \oplus_1 N_1 = \begin{bmatrix} 8 & 1 & 0.5 & 0 & 0 \\ 0.5 & 7 & 1 & 0 & 0 \\ 0.5 & 0.5 & 12 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 9 & 1 \\ 0 & 0 & 1 & 1 & 8 \end{bmatrix} \quad (31)$$

For $i \in \alpha = \{1,2\}, |(p_1)_{ii}| > r_i^{\alpha}(P_1)$,

$(p_1)_{11} = 8 > r_1^{\alpha}(p_1) = |(p_1)_{12}| + |(p_1)_{13}| = 1.5$, $(p_1)_{22} = 7 > r_2^{\alpha}(P_1) = |(p_1)_{21}| + |(p_1)_{23}| = 1.5$.

For $i \in \alpha, j \in \beta \cup \gamma$

$$\left(|(p_1)_{ii}| - r_1^{\alpha}(P_1)\right)\left(|(p_1)_{33}| - r_3^{\beta\cup\gamma}(P_1)\right) = 71.5 > r_3^{\beta\cap\gamma}(P_1).r_3^{\alpha}(P_1) = 1 \quad (32)$$

All the conditions of theorem 1 are satisfied, hence P is α-SDD and nonsingular.

Confirmation that the Σ-strict diagonal dominance property is preserved through k-subdirect sums. Generalization of known results for Σ-SDD matrices to include partial overlap cases. The resulting matrix P is nonsingular, enabling its use in linear systems.

Theorem 2: [2] Let $M \in C^{n_1,n_1}, N \in C^{n_2,n_2}$, with $n_1 \geq 2, and\ 1 \leq \min(n_1, n_2)$. Define the index sets α, β, γ as in (5). For M and N partitioned as in (3), where M_{22} , $N_{11} \in C^{k\times k}$, the corresponding diagonal entries of M_{22} and N_{11} are assumed to have the same sign pattern. For an arbitrary index set $\Sigma \subseteq \{1,2,\dots,n\}$, where $n = n_1 + n_2 - k$, define as $\Sigma_M = \Sigma \cap (\alpha \cup \beta)$ and $\Sigma_N = i - t : i \in \Sigma \cap (\beta \cup \gamma)$. If the following conditions hold:

1. M is an Σ_M-SDD matrix,
2. N is an Σ_N-SDD matrix,
3. $J_M(\Sigma_M) \cap J_N(\Sigma_N) \neq \emptyset$.

then the k-subdirect sum $P = M \oplus_k N$ is an Σ-SDD matrix.

Proof. Choose an index $x \in J_M(\Sigma_M) \cap J_N(\Sigma_N)$ (this intersection is nonempty by assumption). Construct the scaling matrices $X_{n_1}(\Sigma_M, x)\ and\ X_{n_2}(\Sigma_N, x)$. Since M is Σ_M-SDD and N is Σ_N-SDD, the scaled matrices $MX_{n_1}(\Sigma_M, x)\ and\ NX_{n_2}(\Sigma_N, x)$ are strictly diagonally dominant (SDD).

Now, form the matrix $X_n(\Sigma, x)$ It can be verified that:

$$PX_n(\Sigma, x) = M\,X_{n_1}(\Sigma_M, x) \oplus_k N\,X_{n_2}(\Sigma_N, x) \quad (33)$$

Since the k-subdirect sum of two SDD matrices with the diagonal entries of the overlapping blocks M_{22} and N_{22} having the same sign pattern is again an SDD matrix, it follows that $PX_n(\Sigma, x)$ is SDD. Therefore, P is Σ -SDD**.**

To demonstrate the practical necessity of the conditions established in theorem 2, we construct the following numerical illustration. This case examines how the k- subdirect sum fails to preserve the Σ -SDD property when the scaling interval requirement is violated.

Example 2 Consider the matrices M_2 and N_2 defined with the following parameters:

$$M_2 = N_2 = \begin{bmatrix} 2.0 & 0.8 & 0.6 \\ 0.7 & 1.5 & 0.9 \\ 0.5 & 0.8 & 2.2 \end{bmatrix} \quad (34)$$

From a structural standpoint, M_2 and N_2 are both $\{1,2\}$-SDD matrices and $\{3\}$-SDD matrices. To observe how these properties interact during matrix fusion, we construct their 2- subdirect sum, $P_2 = M_2 \oplus_2 N_2$, which results in the following:

$$P_2 = \begin{bmatrix} 2.0 & 0.8 & 0.6 & 0 & 0 \\ 0.7 & 1.5 & 0.9 & 0 & 0 \\ 0.5 & 0.8 & 4.2 & 0.8 & 0.6 \\ 0 & 0 & 0.7 & 1.5 & 0.9 \\ 0 & 0 & 0.5 & 0.8 & 2.2 \end{bmatrix} \quad (35)$$

To diagnose the failure of P_2 to inherit the Σ -SDD property, we take $\Sigma = \{3\}$. Based on the framework of theorem 2, the corresponding parameters are:

1. $\Sigma_M = \Sigma \cap (\alpha \cup \beta) = \{3\}$,
2. $\Sigma_N = i - t : i \in \Sigma \cap (\beta \cup \gamma) = 1$, where $t = n_1 - k = 3 - 1 = 2$.

Our calculations confirm that the scaling intervals are $J_M(\Sigma_M)$ = (0.75,1.25), and $J_N(\Sigma_N)$ = (1.6,2.4). it is evident that $J_M(\Sigma_M) \cap J_N(\Sigma_N) = \emptyset$.

Hence, the subdirect sum P_2 fails to be Σ-SDD due to the empty intersection of the scaling intervals, demonstrating that the subdirect sum of two Σ-SDD matrices is not necessarily Σ-SDD.

Final, the subdirect sum $P_2 = M_2 \oplus_2 N_2$ is Σ-SDD (specifically SDD) despite non-overlapping intervals $J_{M_1}(\Sigma_{M_1})\ \text{and}\ J_{N_1}(\Sigma_{N_1})$, which demonstrates that the sufficient condition of theorem 2 is not necessary.

Applications of Σ -SDD matrices:

You might wonder why we study these matrices. Well, the Σ -SDD property is actually very helpful in many real life tasks. First in Electricity These matrices use to map out power grids because of the special properties we discussed, computers can sole these problems faster. And Σ-SDD use in Engineering When we combine different systems, we must make sure the whole thing stays stable. Adding these matrices helps us know that the system won't fail. Finally in business and economics When mix date from different markets we are basically adding matrices keeping this property ensures our economic models don't give us wrong or crazy results.

Conclusion:

This research has extended the theory of *S*-strictly diagonally dominant matrices to the more general class of Σ-SDD matrices. We have provided complete characterizations for when the *k*-subdirect sum and conventional sum of Σ-SDD matrices preserve the Σ-SDD property. The results contribute to the understanding of matrix structures and their preservation under various operations, with potential applications in numerical linear algebra and computational mathematics.

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