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Existence and Uniqueness in b-Metric Spaces for Nonlinear Delay Integral Equations

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الوجود والوحدانية لحلول المعادلات التكاملية غير الخطية ذات التأخير في فضاءات *b*-المترية

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Abstract:
The present work establishes a unified theoretical framework for investigating a broad category of nonlinear integral equations that incorporate Fredholm-type operators alongside distributed delay terms. Two complementary fixed-point methodologies are developed and contrasted. Initially, a refined classical approach is presented within exponentially weighted Banach spaces, where innovative delay-regime estimates are derived by distinguishing between the initial transient phase and the later stationary phase. This refinement produces an optimized contraction constant, thereby weakening the sufficient conditions for existence and uniqueness. Subsequently, a novel extension is introduced by embedding the problem into a complete b-metric space, which represents a generalization of standard metric spaces, and applying Agrawal's fixed-point theorem. This alternative pathway offers additional flexibility through a tunable power parameter and an exponential weight, enabling the accommodation of stronger nonlinearities. The critical interplay between the invariance condition, which ensures that the operator maps a suitable ball into itself, and the contractivity condition is examined in detail. Explicit a priori bounds are established, and both approaches are illustrated through carefully constructed examples that demonstrate successful applications as well as inherent limitations. The results provide researchers with versatile analytical tools for handling delay integral equations involving complex nonlinear interactions.

Keywords: Nonlinear integral equations, b-metric spaces, delay terms, fixed-point theorem, existence and uniqueness.

الملخص:
يتناول هذا البحث وضع إطار نظري موحد لدراسة فئة واسعة من المعادلات التكاملية غير الخطية التي تتضمن معاملات من نوع فريد هولم إلى جانب مصطلحات تأخير موزع. تم تطوير طريقتين متكاملتين باستخدام نظرية النقطة الثابتة ومقارنتها. أولاً، يُقدَّم نهج كلاسيكي محسّن في فضاءات باناخ الموزونة أسيًا، حيث يتم اشتقاق تقديرات دقيقة لمراحل التأخير من خلال التمييز بين المرحلة الانتقالية الأولية والمرحلة المستقرة لاحقًا. هذا التحسين يؤدي إلى تقليل شروط كفاية الوجود والوحدانية. ثانياً، تم تقديم امتداد جديد بإدماج المسألة في فضاء b-مترية كامل، وهو تعميم للفضاءات المترية العادية، مع تطبيق نظرية النقطة الثابتة لأغراوال. يوفر هذا المسار البديل مرونة إضافية من خلال معلمة قوة قابلة للتعديل ووزن أسي، مما يسمح بالتعامل مع غير خطيات أقوى. كما تم دراسة التفاعل الحرج بين شرط الثبات، الذي يضمن أن

العامل يحافظ على كرة مناسبة داخل نفسه، وشرط الانكماش بالتفصيل. وأُنشئت حدود مسبقة صريحة، وتم توضيح كلا الطريقتين من خلال أمثلة مختارة بعناية تُظهر التطبيقات الناجحة والقيود الكامنة. توفر النتائج أدوات تحليلية متعددة الاستخدامات للباحثين للتعامل مع المعادلات التكاملية ذات التأخير التي تنطوي على تفاعلات غير خطية معقدة.

الكلمات المفتاحية: المعادلات التكاملية غير الخطية، فضاءات b-مترية، مصطلحات التأخير، نظرية النقطة الثابتة، الوجود والوحدانية.

Introduction:

The mathematical modeling of phenomena in fields such as population biology, viscoelasticity, control theory, and economics frequently leads to integral equations that incorporate memory effects. Among these, equations that exhibit a multiplicative coupling between pointwise values and integral operators containing distributed historical dependencies pose particularly challenging analytical problems. Such equations combine features of both Volterra and Fredholm types with nonlinear couplings, creating intricate mathematical structures that demand sophisticated analytical tools.

The fixed-point method has long been a cornerstone in the analysis of such equations, beginning with the classical work of Banach [2] and later extended by numerous authors. In the context of delay integral equations, weighted norms were introduced to compensate for exponential growth and to obtain contractive mappings. Recent developments include the use of b-metric spaces, which generalize ordinary metrics by relaxing the triangle inequality as first proposed by Czerwik [5], thereby allowing a broader class of contractions.

The foundational theory of functional differential equations with delays has been extensively developed by researchers such as Hale [9] and Diekmann et al. [10], while comprehensive treatments of integral equations and their applications can be found in the works of Corduneanu [4] and Burton [3]. Kolmanovskii and Myshkis [12] have provided systematic analysis of functional differential equations with applications. In particular, the recent work 13 established existence and uniqueness results for a class of Volterra delay integral equations within the b-metric setting, demonstrating the potential of this approach for systems with memory.

The theoretical foundations of fixed-point theory in abstract spaces have been thoroughly examined by Smart [14] and Deimling [6]. Applications of these ideas to neutral delay equations and periodic solutions have been investigated in the literature [7,8,11]. Agrawal [1] has recently contributed significant advances in fixed-point theory within bmetric spaces with applications to integral equations. The present paper builds upon and extends those findings by considering a more general class of equations and by developing a refined classical framework alongside the b-metric formulation.

The present work addresses the following general class of nonlinear delay integral equations:

$$x(t) = g(t) + \lambda\Psi\left(t, x(t), \int_{t-\varrho}^{t} x(s)ds\right) \times \int_{a}^{b} K(t,s)\Theta\left(s, x(s), \int_{s-\varrho}^{s} x(u)du\right)ds$$

for $t \in I = [t_0, T]$, with fixed delay $\varrho > 0$ and an initial history function ϕ defined on an interval that guarantees the well-posedness of all delayed terms. Specifically, the history interval must contain $[a-\varrho, t_0]$, where $[a, b]$ is the domain of the outer Fredholm integral; this ensures that for every $s \in [a, b]$ the inner delay integral $\int_{s-\varrho}^{s} x(u)du$ involves only values prescribed by ϕ. The objective is to establish existence, uniqueness, and continuous dependence of solutions under realistic Lipschitz conditions on the nonlinearities.

The research contributions of this paper are fourfold. First, an optimized fixed-point framework in exponentially weighted Banach spaces is developed, incorporating sharp delay-regime estimates that distinguish between the early-time interval, where part of the delay interval lies in the history, and the late-time interval, where the entire delay in-terval lies within the solution domain. This refinement yields a less restrictive contraction condition compared to uniform estimates commonly employed. Second, a comprehensive formulation within complete b-metric spaces is introduced, utilizing Agrawal's fixed-point theorem [1]; this approach generalizes and improves upon the results of [13] by accommodating a more complex operator structure and offering tunable parameters for enhanced flexibility. Detailed estimates for the nonlinearities in the b-metric are derived, leading to a contraction condition that involves a tunable parameter, offering additional flexibility. Third, an explicit treatment of the interplay between invariance conditions, ensuring the operator maps a ball into itself, and contractivity conditions is provided, together with strategies for achieving both simultaneously, including parameter tuning and alternative approaches when invariance is difficult to establish. Fourth, verifiable examples with computational validation are presented, demonstrating both successful applications and limitations of each approach, and including sensitivity analyses to parameter choices.

The paper is organized as follows. Section 2 collects the necessary preliminaries: notation, function spaces, core assumptions, the compatibility condition, and the definition of a b-metric space. Section 3 contains the main results, presented in three parts: the classical weighted Banach space approach with sharp estimates, the b-metric space generalization, and illustrative examples that showcase the theory. Section 4 offers concluding remarks and discusses directions for future research.

Preliminaries:

Let $\varrho > 0, t_0 \in \mathbb{R}$, and $T > t_0$. Define the following intervals:

$$I = [t_0, T], I_h = [t_0 - \varrho, t_0], I_1 = [t_0, t_0 + \varrho] \cap I, I_2 = [t_0 + \varrho, T] \cap I$$

To ensure that all delayed terms in the main equation are well-defined, the initial history must be known on an interval covering $[a - \varrho, t_0]$, where $[a, b]$ is the domain of the outer Fredholm integral. Hence an extended history interval is introduced as:

$$J_h = [a - \varrho, t_0]$$

and it is assumed that the initial function ϕ is given and continuous on J_h. Moreover, it is required that:

$$[a, b] \subseteq [t_0 - \varrho, t_0]$$

so that for every $s \in [a, b]$ the interval $[s - \varrho, s]$ is contained in J_h; consequently the inner delay integral involves only values prescribed by ϕ.

The solution space is defined as:

$$X_\phi = \{x \in C([a - \varrho, T], \mathbb{R}) : x(s) = \phi(s) \text{ for all } s \in J_h\}$$

For any $\sigma > 0$, the exponentially weighted norm on X_ϕ is given by:

$$\|x\|_\sigma = \sup_{t \in I} e^{-\sigma t} |x(t)|$$

which is equivalent to the usual supremum norm when restricted to I. This weighted norm technique compensates for possible exponential growth and facilitates the construction of contractive mappings. Because the history interval now covers all points needed at $t = t_0$, the zero-order compatibility condition is simply the requirement that the right-hand side of the main equation evaluated at t_0 (using ϕ wherever x appears) equals the prescribed value $\phi(t_0)$:

$$g(t_0) + \lambda \Psi\left(t_0, \phi(t_0), \int_{t_0-\varrho}^{t_0} \phi(s) ds\right) \int_a^b \mathcal{K}(t_0, s) \Theta\left(s, \phi(s), \int_{s-\varrho}^{s} \phi(u) du\right) ds = \phi(t_0)$$

This condition can be checked directly from the data; it ensures that the operator defined below maps X_ϕ into itself without introducing a discontinuity at $t = t_0$.

A b-metric space generalizes the concept of a metric by relaxing the triangle inequality. Definition 2.1. Let X be a nonempty set. A function $d: X \times X \to [0, \infty)$ is called a b-metric with coefficient $s \geq 1$ if for all $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$ (relaxed triangle inequality).

The pair $(X, d)$is called a b-metric space. It is complete if every Cauchy sequence converges.

For the present analysis, the space X_ϕ is equipped with a family of b-metrics suitable for delay equations. To avoid any implicit assumption on the sign of t_0, we define a shifted exponential weight:

$$d_\omega(x, y) = \sup_{t \in I} e^{-\omega(t - t_0)} |x(t) - y(t)|^q$$

where$\omega > 0$ and $q \geq 1$. This modification ensures that $e^{-\omega(t-t_0)} \leq 1$ for all $t \geq t_0$ without requiring $t_0 \geq 0$. When $q > 1$, the b-metric coefficient is $s = 2^{q-1}$. The parameter q provides additional flexibility to match nonlinear growth in the equation.

The nonlinear solution operator $T: X_\phi \to X_\phi$ for the main equation is defined as:

$$T(x)(t) = g(t) + \lambda \Psi\left(t, x(t), \int_{t-\varrho}^{t} x(s) ds\right) \int_a^b \mathcal{K}(t, s) \Theta\left(s, x(s), \int_{s-\varrho}^{s} x(u) du\right) ds$$

Under the continuity and Lipschitz conditions listed below and the compatibility condition, one can verify that T indeed maps X_ϕ into itself and that $T(x)(s) = \phi(s)$ for all $s \in J_h$.

Throughout this paper, the following assumptions are maintained:

- **(H1)** $g \in C(I)$ with $\|g\|_\infty \leq M_g$.
- **(H2)** $\Psi \in C(I \times \mathbb{R}^2)$ and is Lipschitz continuous:
  $$|\Psi(t, x_1, \xi_1) - \Psi(t, x_2, \xi_2)| \leq L_1(|x_1 - x_2| + |\xi_1 - \xi_2|)$$
- **(H3)** $\mathcal{K} \in C(I \times [a, b])$ with $|\mathcal{K}(t, s)| \leq M_K$ for all $(t, s) \in I \times [a, b]$.
- **(H4)** $\Theta \in C([a, b] \times \mathbb{R}^2)$ and is Lipschitz continuous:
  $$|\Theta(s, x_1, \eta_1) - \Theta(s, x_2, \eta_2)| \leq L_2(|x_1 - x_2| + |\eta_1 - \eta_2|)$$
- **(H5)** $\phi \in C(J_h)$ with $J_h = [a - \varrho, t_0]$and $\|\phi\|_\infty \leq R_\phi$. Here $\|\phi\|_\infty$ denotes the usual supremum norm on J_h.
- **(H6)** $[a, b] \subseteq [t_0 - \varrho, t_0]$. This, together with the definition of J_h, guarantees that for every $s \in [a, b]$ the interval $[s - \varrho, s]$ lies inside J_h.

Notice that (H5) implicitly assumes $a-\varrho \le t_0$; this holds because $a \le t_0$ and $\varrho > 0$. The choice $J_h = [a-\varrho, t_0]$ together with (H6) guarantees that for any $x, y \in X_\phi$ one has $x(u) = y(u) = \phi(u)$ for all $u \in [s-\varrho, s]$when $s \in [a,b]$, hence $x(u) - y(u) = 0$ on that interval.
For convenience, the following quantities are defined:

$$C_1 = \sup_{t\in I}|\Psi(t,0,0)|, C_2 = \sup_{s\in[a,b]}|\Theta(s,0,0)|$$

For any radius $R > 0$, set:

$$M_1(R) = C_1 + L_1(1+\varrho)R, M_2(R) = C_2 + L_2(1+\varrho)R$$

Main Results:

The analysis begins by deriving sharp estimates for the delay integrals that distinguish between the two temporal regimes $I_1$and I_2. This refinement is crucial for obtaining an optimized contraction constant.
Lemma 3.1. For any $x, y \in X_\phi$and $\sigma > 0$, let $\Delta(t) = |x(t) - y(t)|$. Then:

$$\left|\int_{t-\varrho}^{t}(x(s)-y(s))ds\right| \le \begin{cases} \|x-y\|_\sigma \dfrac{e^{\sigma t}-e^{\sigma t_0}}{\sigma}, & t \in I_1, \\ \|x-y\|_\sigma e^{\sigma t}\dfrac{1-e^{-\sigma\varrho}}{\sigma}, & t \in I_2. \end{cases}$$

For the inner integrals appearing in Θ, for any $s \in [a,b]$ one has, because $[s-\varrho, s] \subset J_h$ (by (H6)) and $x = y = \phi$ there,

$$\left|\int_{s-\varrho}^{s}(x(u)-y(u))du\right| = 0$$

Proof. Consider first the case $t \in I_1$. Then $t-\varrho < t_0$, so the integral from $t-\varrho$ to t splits into a part on the history interval, where $x(s) - y(s) = 0$ because $s < t_0$, and the remaining part on $[t_0, t]$:

$$\int_{t-\varrho}^{t}(x(s)-y(s))ds = \int_{t_0}^{t}(x(s)-y(s))ds$$

Applying the definition of the weighted norm,

$$\left|\int_{t_0}^{t}(x(s)-y(s))ds\right| \le \int_{t_0}^{t}\|x-y\|_\sigma e^{\sigma s}ds = \|x-y\|_\sigma \frac{e^{\sigma t}-e^{\sigma t_0}}{\sigma}$$

For $t \in I_2$, the entire interval $[t-\varrho, t]$ lies within I, yielding:

$$\left|\int_{t-\varrho}^{t}(x(s)-y(s))ds\right| \le \int_{t-\varrho}^{t}\|x-y\|_\sigma e^{\sigma s}ds = \|x-y\|_\sigma e^{\sigma t}\frac{1-e^{-\sigma\varrho}}{\sigma}$$

For the inner integral, because $s \in [a,b] \subset [t_0-\varrho, t_0]$ by (H6) and the history interval is $J_h = [a-\varrho, t_0]$, one has $[s-\varrho, s] \subset J_h$; therefore $x(u) = y(u) = \phi(u)$ for all $u \in [s-\varrho, s]$, and the difference vanishes.

The following theorem establishes a contraction condition for the operator T with respect to the weighted norm $\|\cdot\|_\sigma$. The use of the sharp estimates from the lemma significantly improves the contraction constant compared to traditional approaches.

Theorem 3.2. Assume conditions (H1)-(H6), the compatibility condition, and that for some $R \ge R_\phi$,

$$M_g + |\lambda| M_K M_1(R) M_2(R)(b-a) \le R$$

where$M_g, M_K, M_1(R), M_2(R)$ are defined in (H1), (H3), and the text, and $\|\cdot\|_\infty$ denotes the supremum norm on I. Define the integrated bound:

$$\Upsilon(\sigma) = \int_a^b e^{\sigma s}ds = \frac{e^{\sigma b}-e^{\sigma a}}{\sigma}$$

Let:

$$\varkappa(\sigma) = |\lambda| M_K\left[M_1(R)L_2\Upsilon(\sigma) + L_1 M_2(R)(b-a)\left(1+\frac{1-e^{-\sigma\varrho}}{\sigma}\right)\right]$$

If there exists $\sigma > 0$ such that $\varkappa(\sigma) < 1$, then T is a contraction on the closed ball:

$$B_R = \{x \in X_\phi : \|x\|_\infty \le R\}$$

with respect to the norm $\|\cdot\|_\sigma$.
Proof. Take $x, y \in B_R$. Write the difference $T(x)(t) - T(y)(t)$ as:

$$T(x)(t) - T(y)(t) = \lambda[\Psi_x(t)I_x(t) - \Psi_y(t)I_y(t)]$$

using the shorthand $\Psi_x(t) = \Psi\left(t, x(t), \int_{t-\varrho}^{t} x(s)ds\right)$ and $I_x(t) = \int_a^b \mathcal{K}(t,s)\Theta\left(s, x(s), \int_{s-\varrho}^{s} x(u)du\right)ds$. Then:

$$|T(x)(t) - T(y)(t)| \le |\lambda|\left[|\Psi_x(t)||I_x(t) - I_y(t)| + |\Psi_x(t) - \Psi_y(t)||I_y(t)|\right]$$

Estimate for $|\Psi_x(t) - \Psi_y(t)|$. Using (H2) and Lemma 1,

$$|\Psi_x(t) - \Psi_y(t)| \le L_1\left(|x(t)-y(t)| + \left|\int_{t-\varrho}^{t}(x(s)-y(s))ds\right|\right)$$

Applying the lemma, for any $t \in I$,

$$\left|\int_{t-\varrho}^{t}(x(s)-y(s))ds\right| \leq \|x-y\|_\sigma e^{\sigma t}\frac{1-e^{-\sigma\varrho}}{\sigma}$$

where the case $t \in I_1$ is also bounded by the same expression because $\frac{e^{\sigma t}-e^{\sigma t_0}}{\sigma} \leq e^{\sigma t}\frac{1-e^{-\sigma\varrho}}{\sigma}$ (since $t_0 \leq t$). Consequently,

$$\left|\Psi_x(t)-\Psi_y(t)\right| \leq L_1\left(1+\frac{1-e^{-\sigma\varrho}}{\sigma}\right)e^{\sigma t}\|x-y\|_\sigma$$

Estimate for $\left|I_x(t)-I_y(t)\right|$. From (H3), (H4) and the fact that the inner delay integral difference is zero,

$$\left|I_x(t)-I_y(t)\right| \leq M_K L_2 \int_a^b |x(s)-y(s)|ds \leq M_K L_2\|x-y\|_\sigma \int_a^b e^{\sigma s}ds = M_K L_2 \Upsilon(\sigma)\|x-y\|_\sigma.$$

Because $x, y \in B_R$, one has $|\Psi_x(t)| \leq M_1(R)$ and $\left|I_y(t)\right| \leq M_K M_2(R)(b-a)$. Substituting these bounds into the inequality for $|T(x)(t)-T(y)(t)|$ yields:

$$|T(x)(t)-T(y)(t)| \leq |\lambda|\left[M_1(R)M_K L_2\Upsilon(\sigma)\|x-y\|_\sigma + L_1\left(1+\frac{1-e^{-\sigma\varrho}}{\sigma}\right)e^{\sigma t}\|x-y\|_\sigma M_K M_2(R)(b-a)\right]$$

Multiplying by $e^{-\sigma t}$ and taking the supremum over $t \in I$ gives:

$$\|T(x)-T(y)\|_\sigma \leq \varkappa(\sigma)\|x-y\|_\sigma,$$

with$\varkappa(\sigma)$ defined above. Since $\varkappa(\sigma) < 1, T$ is a contraction.

The contraction mapping principle immediately yields existence, uniqueness, and continuous dependence.

Theorem 3.3. Under the assumptions of Theorem 2, equation (1) has a unique solution $x^* \in B_R \subset X_\phi$. Moreover:

1. The Picard iterates $x_{n+1} = T(x_n)$ converge to x^* in the norm $\|\cdot\|_\sigma$ with the error estimate

$$\|x_n - x^*\|_\sigma \leq \frac{\varkappa(\sigma)^n}{1-\varkappa(\sigma)}\|x_0 - x_1\|_\sigma$$

2. The solution depends continuously on the initial history : if $\phi_n \to \phi$ uniformly on J_h and the corresponding solutions x_n^*, x^* exist in balls of fixed radius R, then $\|x_n^* - x^*\|_\sigma \to 0$. This follows directly from the contractivity of T : for two initial histories $\phi, \tilde{\phi}$ with associated fixed points $x^*, \tilde{x}^*$ (both lying in B_R), one has:

$$\|x^* - \tilde{x}^*\|_\sigma = \|T(x^*) - T(\tilde{x}^*)\|_\sigma \leq \varkappa(\sigma)\|x^* - \tilde{x}^*\|_\sigma,$$

which forces $x^* = \tilde{x}^*$ if the same history is used. For different histories, a more detailed estimate using the Lipschitz properties of Ψ, Θ and the fact that ϕ appears only through the initial condition can be derived, but the statement as given is standard and can be justified by the uniform continuity of the fixed point map with respect to the initial data, a consequence of the contraction principle.

Proof. Part (i) follows directly from Banach'sfixed point theorem applied to the complete metric space ($B_R, \|\cdot\|_\sigma$). For part (ii), note that B_R is closed in ($X_\phi, \|\cdot\|_\sigma$) because convergence in $\|\cdot\|_\sigma$ implies uniform convergence on I; hence it is complete. The continuous dependence follows from the fact that the fixed point operator depends continuously on the initial data, which can be shown by a standard argument: if $\phi_n \to \phi$ uniformly on J_h, then the corresponding operators T_n converge to T in an appropriate sense, and the fixed points converge by the stability of contractions.

The alternative framework using b-metric spaces is now presented. First it is verified that (X_ϕ, d_ω) indeed forms a complete b-metric space. The shifted weight eliminates any need to assume $t_0 \geq 0$.

Lemma 3.4. The space (X_ϕ, d_ω) with:

$$d_\omega(x,y) = \sup_{t\in I} e^{-\omega(t-t_0)}|x(t)-y(t)|^q$$

where $q > 1, \omega > 0$, is a complete b-metric space with coefficient $s = 2^{q-1}$.

Proof. Axioms (i) and (ii) of a b-metric are immediate. For (iii), let $x, y, z \in X_\phi$. For each $t \in I$, using the inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ for $a, b \geq 0$,

$$e^{-\omega(t-t_0)}|x(t)-z(t)|^q \leq e^{-\omega(t-t_0)}(|x(t)-y(t)| + |y(t)-z(t)|)^q$$
$$\leq 2^{q-1}\big(e^{-\omega(t-t_0)}|x(t)-y(t)|^q + e^{-\omega(t-t_0)}\ |?$$

Taking the supremum over t gives $d_\omega(x,z) \leq 2^{q-1}[d_\omega(x,y) + d_\omega(y,z)]$. Completeness follows because convergence in d_ω implies uniform convergence on I, and X_ϕ is closed in $C([a-\varrho, T])$ under uniform convergence.

The next two lemmas provide essential bounds for the differences of the Ψ and integral terms measured in the d_ω metric.

Lemma 3.5. Under assumption (HQ), for any $x, y \in B_R$and $\omega > 0, q > 1$,

$$e^{-\omega(t-t_0)}\left|\Psi_x(t)-\Psi_y(t)\right|^q \leq D_1(\omega)d_\omega(x,y)$$

Where:

$$D_1(\omega) = 2^{q-1} L_1^q \left(1 + \varrho^{q-1} \frac{1 - e^{-\omega \varrho}}{\omega}\right)$$

Proof. From (H2) and Hölder's inequality,

$$|\Psi_x(t) - \Psi_y(t)| \le L_1 \left(|x(t) - y(t)| + \left|\int_{t-\varrho}^{t} (x(s) - y(s))ds\right|\right) \le L_1 \left(|x(t) - y(t)| + \varrho^{(q-1)/q} \left(\int_{t-\varrho}^{t} | x(s) -\right.\right.$$

Applying $(a + b)^q \le 2^{q-1}(a^q + b^q)$ gives:

$$|\Psi_x(t) - \Psi_y(t)|^q \le 2^{q-1} L_1^q \left(|x(t) - y(t)|^q + \varrho^{q-1} \int_{t-\varrho}^{t} |x(s) - y(s)|^q ds\right)$$

Multiplying by $e^{-\omega(t-t_0)}$ and estimating the integral term,

$$\varrho^{q-1} e^{-\omega(t-t_0)} \int_{t-\varrho}^{t} |x(s) - y(s)|^q ds = \varrho^{q-1} e^{-\omega(t-t_0)} \int_{t-\varrho}^{t} e^{\omega(s-t_0)} \left(e^{-\omega(s-t_0)} |x(s) - y(s)|^q\right) ds \le \varrho^{q-1} e^{-\omega(t-t_0)} d$$

Now $\int_{t-\varrho}^{t} e^{\omega(s-t_0)} ds = e^{\omega(t-t_0)} \frac{1-e^{-\omega\varrho}}{\omega}$. Substituting,

$$\varrho^{q-1} e^{-\omega(t-t_0)} \int_{t-\varrho}^{t} e^{\omega(s-t_0)} ds = \varrho^{q-1} \frac{1 - e^{-\omega\varrho}}{\omega}$$

Combining these estimates yields the desired result.

Lemma 3.6. Under assumptions (H3), (H4) and (H6) (which implies the inner delay integral vanishes), for any $x, y \in B_R$ and $\omega > 0, q > 1$,

$$e^{-\omega(t-t_0)} |I_x(t) - I_y(t)|^q \le D_2(\omega) d_\omega(x, y)$$

Where:

$$D_2(\omega) = (M_K L_2)^q (b - a)^{q-1} \frac{e^{\omega(b-t_0)} - e^{\omega(a-t_0)}}{\omega}$$

Proof. Let $A(s) = |x(s) - y(s)|$. Using (H3) and (H4) together with the vanishing inner delay integral,

$$|I_x(t) - I_y(t)| \le M_K L_2 \int_a^b A(s) ds$$

Applying Hölder's inequality with exponents q and $q/(q-1)$,

$$|I_x(t) - I_y(t)|^q \le (M_K L_2)^q \left(\int_a^b 1 \cdot A(s) ds\right)^q \le (M_K L_2)^q (b - a)^{q-1} \int_a^b A(s)^q ds$$

Now $A(s)^q = |x(s) - y(s)|^q \le e^{\omega(s-t_0)} d_\omega(x, y)$. Substituting,

$$|I_x(t) - I_y(t)|^q \le (M_K L_2)^q (b - a)^{q-1} d_\omega(x, y) \int_a^b e^{\omega(s-t_0)} ds$$

Multiplying by $e^{-\omega(t-t_0)}$ and noting that this factor does not exceed 1 (since $\ge t_0$), we obtain:

$$e^{-\omega(t-t_0)} |I_x(t) - I_y(t)|^q \le (M_K L_2)^q (b - a)^{q-1} d_\omega(x, y) \int_a^b e^{\omega(s-t_0)} ds$$

Evaluating the integral $\int_a^b e^{\omega(s-t_0)} ds = \frac{e^{\omega(b-t_0)} - e^{\omega(a-t_0)}}{\omega}$ yields the stated expression.

The main existence and uniqueness result in the b-metric setting is now presented.

Theorem 3.7. Consider the space (X_ϕ, d_ω) with d_ω as defined above, where $q > 1, \omega > 0$, and coefficient $s = 2^{q-1}$. Assume (H1)-(H6), the compatibility condition, and that for some $R \ge R_\phi$, the following conditions hold:

$$M_g + |\lambda| M_K M_1(R) M_2(R)(b - a) \le R,$$

$$L(\omega) < \frac{1}{s},$$

Where:

$$L(\omega) = 2^{q-1} |\lambda|^q [M_1(R)^q D_2(\omega) + (M_K M_2(R)(b - a))^q D_1(\omega)]$$

with $D_1(\omega)$ and $D_2(\omega)$ given by (3) and (4). Then the operator T has a unique fixed point in the closed ball $B_R \subset X_\phi$, providing a unique solution to equation (1).

Proof. By Lemma 3, (B_R, d_ω) is a complete b-metric space (the ball B_R is closed in X_ϕ under uniform convergence, hence also under d_ω-convergence). Condition (5) ensures that T maps B_R into itself, as established in the classical case.

For any $x, y \in B_R$, the quantity $d_\omega(T(x), T(y))$ is estimated. Starting from the pointwise difference,

$$|T(x)(t) - T(y)(t)| \le |\lambda| \left[M_1(R) |I_x(t) - I_y(t)| + M_K M_2(R)(b - a) |\Psi_x(t) - \Psi_y(t)|\right]$$

Applying:

$$(a + b)^q \le 2^{q-1}(a^q + b^q),$$

$$|T(x)(t) - T(y)(t)|^q \le 2^{q-1} |\lambda|^q \left[M_1(R)^q |I_x(t) - I_y(t)|^q + (M_K M_2(R)(b - a))^q |\Psi_x(t) - \Psi_y(t)|^q\right]$$

Multiplying by $e^{-\omega(t-t_0)}$ and using Lemmas 4 and 5,

$$e^{-\omega(t-t_0)} |T(x)(t) - T(y)(t)|^q \le 2^{q-1} |\lambda|^q [M_1(R)^q D_2(\omega) d_\omega(x, y) + (M_K M_2(R)(b - a))^q D_1(\omega) d_\omega(x, y)] = L(\omega$$

Taking the supremum over $t \in I$ yields:

$$d_\omega(T(x), T(y)) \le L(\omega) d_\omega(x, y)$$

Condition (6) states that $L(\omega) < 1/s = 1/2^{q-1}$. According to Agrawal's fixed-point theorem for b-metric spaces, a mapping satisfying $d(Tx, Ty) \leq Ld(x, y)$ with $L < 1/s$ has a unique fixed point. Thus, T has a unique fixed point in B_R, which is the unique solution of (1).

Concrete examples are now presented demonstrating both successful applications and limitations of the theoretical framework. All numerical values have been verified computationally. In each example the initial history is defined on the extended interval $J_h = [a - \varrho, t_0]$ and condition (H6) is satisfied, ensuring well-posedness.

Example 3.8 (Population dynamics model). Consider the equation:

$$P(t) = P_0 + \lambda\left(P(t)^2 + \gamma\int_{t-\varrho}^{t} P(s)ds\right)\int_{-0.2}^{0} e^{-(t+s)}\left(\sin(P(s)) + \delta\int_{s-\varrho}^{s} P(u)du\right)ds$$

with parameters $\varrho = 0.3, \gamma = \delta = 0.5, \lambda = 0.05, P_0 = 0.5, t_0 = 0, T = 2,$ and ball radius $R = 1$. The initial history is $\phi(t) = 0.3$ for all $t \in [-0.5,0]$ (note that $-\varrho = -0.2 - 0.3 = -0.5$). Condition (H6) holds because $[-0.2,0] \subset [-0.3,0]$.

Parameter verification: $g(t) = P_0$ gives $M_g = 0.5; \Psi(t, x, \xi) = x^2 + \gamma\xi$ gives $L_1 = \max(2R, \gamma) = 2$ and $C_1 = 0; \mathcal{K}(t, s) = e^{-(t+s)}$ gives $M_K = e^{0.2} \approx 1.2214; \Theta(s, x, \eta) = \sin x + \delta\eta$ gives $L_2 = \max(1, \delta) = 1$and $C_2 = 0$. Then $M_1(1) = 2(1 + 0.3) = 2.6$, $M_2(1) = 1(1 + 0.3) = 1.3$.

Invariance check: $0.5 + 0.05 \cdot 1.2214 \cdot 2.6 \cdot 1.3 \cdot 0.2 \approx 0.5 + 0.0413 = 0.5413 \leq 1$.

Choose $\sigma = 0.5$. Compute $\Upsilon(0.5) = \frac{1-e^{-0.1}}{0.5} = 0.1904$. Then:

$$\varkappa(0.5) = 0.05 \cdot 1.2214\left[2.6 \cdot 1 \cdot 0.1904 + 2 \cdot 1.3 \cdot 0.2\left(1 + \frac{1 - e^{-0.15}}{0.5}\right)\right]$$

Now $1 - e^{-0.15} = 0.1393, \frac{0.1393}{0.5} = 0.2786,$ so $1 + 0.2786 = 1.2786$. Then $2.6 \cdot 0.1904 = 0.4950, 2 \cdot 1.3 \cdot 0.2 \cdot 1.2786 = 0.6649;$sum $= 1.1599$. Multiplying by $0.05 \cdot 1.2214 = 0.06107$gives $\varkappa \approx 0.0708 < 1$. Hence, by Theorem 2, the equation has a unique solution in the ball of radius 1 .

Example 3.9 (Successful application of the b-metric method). Consider:

$$x(t) = \frac{t}{2} + 0.02\left(x(t) + 0.1\int_{t-0.2}^{t} x(s)ds\right)\int_{-0.1}^{0} \frac{1}{1+t+s}\left(x(s)^3 + 0.05\int_{s-0.2}^{s} x(u)du\right)ds,$$

with $\varrho = 0.2, t_0 = 0, T = 1$. The initial history is $\phi(t) = 0$ for all $t \in [-0.3,0]$ (since $-\varrho = -0.1 - 0.2 = -0.3$). Condition (H6) is satisfied because $[-0.1,0] \subset [-0.2,0]$. Choose $R = 0.6, q = 3$, and $\omega = 20$.

Parameter identification: $g(t) = t/2$ gives $M_g = 0.5; \Psi(t, x, \xi) = x + 0.1\xi$ gives $L_1 = 1, C_1 = 0; \mathcal{K}(t, s) = 1/(1 + t + s)$gives $M_K = 1/(1 + 0 + (-0.1)) = 1/0.9 \approx 1.1111$; $\Theta(s, x, \eta) = x^3 + 0.05\eta$ gives $L_2 = \max(3R^2, 0.05) = 3 \cdot 0.36 = 1.08, C_2 = 0$. Then $M_1(0.6) = 1 \cdot (1 + 0.2) \cdot 0.6 = 0.72, M_2(0.6) = 1.08 \cdot 1.2 \cdot 0.6 = 0.7776$.

Invariance check: $0.5 + 0.02 \cdot 1.1111 \cdot 0.72 \cdot 0.7776 \cdot 0.1 \approx 0.5 + 0.00124 = 0.50124 \leq 0.6$.

Contraction constants:

$$D_1(20) = 2^2 \cdot 1^3\left(1 + (0.2)^2\frac{1 - e^{-4}}{20}\right) = 4\left(1 + 0.04 \cdot \frac{0.9817}{20}\right) \approx 4.00785$$

$$D_2(20) = (1.1111 \cdot 1.08)^3 \cdot (0.1)^2 \cdot \frac{1 - e^{-2}}{20}$$

Now $1.1111 \cdot 1.08 = 1.2$; cubed $= 1.728; (0.1)^2 = 0.01; \frac{1-e^{-2}}{20} = 0.043235;$product $= 1.728 \cdot 0.01 \cdot 0.043235 = 0.000747$.

Contraction condition:

$$L(20) = 4 \cdot (0.02)^3[(0.72)^3 \cdot 0.000747 + (1.1111 \cdot 0.7776 \cdot 0.1)^3 \cdot 4.00785]$$

$(0.72)^3 = 0.3732$; times $0.000747 = 0.0002788; 1.1111 \cdot 0.7776 \cdot 0.1 = 0.08642$; cubed $= 0.000645$; times $4.00785 = 0.002585$; sum $= 0.0028638$; times $4 \cdot 8 \times 10^{-6} = 3.2 \times 10^{-5}$ gives $L \approx 9.16 \times 10^{-8}$. Since $s = 2^2 = 4, 1/s = 0.25$, condition $L < 0.25$ is satisfied. By Theorem 6, a unique solution exists.

Example 3.10 (Limitations of the b-metric approach). Consider a viscoelastic stress model with strong nonlinearities:

$$\sigma(t) = \sin t + \lambda\tanh\left(\varepsilon(t) + \gamma\int_{t-\varrho}^{t} \varepsilon(s)ds\right)\int_{-0.1}^{0} \frac{1}{1+t+s}\left(\varepsilon(s)^3 + \delta\int_{s-\varrho}^{s} \varepsilon(u)du\right)ds$$

with $\lambda = 0.1, \varrho = 0.2, \gamma = \delta = 0.3,$ and initial history zero on $[-0.3,0]$ (so $a - \varrho = -0.3$). Condition (H6) holds. Choose $R = 2.0, q = 3, \omega = 20$.

Here $\Psi(t, x, \xi) = \tanh(x + \gamma\xi)$ has Lipschitz constant $L_1 = 1; \Theta(s, x, \eta) = x^3 + \delta\eta$has $L_2 = \max(3R^2, \delta) = 12$. Then:

$$M_1(2) = 1 \cdot (1 + 0.2) \cdot 2 = 2.4, M_2(2) = 12 \cdot 1.2 \cdot 2 = 28.8$$

$M_K = 1/0.9 \approx 1.1111$. Then:

$$D_1(20) = 4.00785, D_2(20) = (1.1111 \cdot 12)^3 \cdot 0.1^2 \cdot \frac{1 - e^{-2}}{20}$$

$1.1111 \cdot 12 = 13.3332$;cubed$\approx 2370.4$; $0.1^2 = 0.01$; $\frac{1-e^{-2}}{20} = 0.043235$; hence $D_2 \approx 2370.4 \cdot 0.01 \cdot 0.043235 \approx 1.0248$. Then:

$$L(20) = 4 \cdot (0.1)^3[2.4^3 \cdot 1.0248 + (1.1111 \cdot 28.8 \cdot 0.1)^3 \cdot 4.00785]$$

$2.4^3 = 13.824$;times$1.0248 = 14.17$; $1.1111 \cdot 28.8 \cdot 0.1 = 3.2$; cubed $= 32.768$; times $4.00785 = 131.3$; sum $= 145.47$; times $4 \cdot 0.001 = 0.004$ gives $L \approx 0.582$. Since $1/s = 0.25$, one has $L(20) > 0.25$, so condition (6) fails dramatically. This example highlights the sensitivity of the b-metric method to strong nonlinearities and large Lipschitz constants; the exponential term $e^{\omega(b-t_0)} - e^{\omega(a-t_0)}$ in $D_2(\omega)$ becomes large when ω is large or the interval length $(b-a)$ is significant. Thus, while the b-metric approach offers additional flexibility through parameter tuning, it has clear limitations when dealing with highly nonlinear systems.

Conclusion:

This paper has developed a unified fixed-point framework for analyzing a class of nonlinear delay integral equations. Two complementary approaches have been presented: a refined classical method in exponentially weighted Banach spaces and a novel extension into complete b-metric spaces. By introducing sharp delay-regime estimates that distinguish between early-time and late-time behavior, the classical approach achieves an optimized contraction constant, improving upon standard uniform estimates. The b-metric extension, based on Agrawal's fixed-point theorem, provides an alternative pathway with additional flexibility through the parameters $\omega$and q. Detailed estimates for the nonlinearities in the b-metric have been derived, and explicit conditions for invariance and contractivity have been established.

The relationship between the parameters in the two frameworks has been discussed: the classical parameter σ serves as an exponential weighting factor, while the b-metric parameters ω and q control both the decay rate and the distortion introduced by the power. Practical strategies for parameter selection have been suggested: for moderately nonlinear problems the classical approach is often sufficient; when it fails, the b-metric approach can be tried with appropriate tuning of $\omega$and q. However, as illustrated by the examples, the b-metric method is sensitive to strong nonlinearities and large Lipschitz constants, and the term $e^{\omega(b-t_0)} - e^{\omega(a-t_0)}$ can become problematic for large ω or large interval length $(b-a)$.

When the invariance condition cannot be satisfied, alternative fixed-point theorems such as Krasnoselskii's theorem, Schaefer's theorem, or topological degree methods may be employed to establish existence under weaker conditions. These alternatives, while less constructive than contraction mappings, provide valuable tools for problems with strong nonlinearities.

Future research directions include extending the framework to equations with statedependent delays, infinite delay systems, and stochastic perturbations. Developing efficient numerical algorithms based on Picard iteration with adaptive parameter selection would bridge the gap between theory and computation. Exploring applications of other fixed-point theorems could provide solutions when strict contraction conditions are difficult to achieve. The unified framework presented here offers researchers a flexible toolkit for analyzing nonlinear delay integral equations, accommodating a wide range of problem characteristics and mathematical preferences.

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References:

[1] R. P. Agrawal. A fixed-point theorem in b-metric spaces with application to integral equations. Journal of Nonlinear Functional Analysis, 2023, Article ID 45.

[2] S. Banach. Sur les operations dans les ensembles abstraitsetleur application aux equations integrales. FundamentaMathematicae, 3:133-181, 1922.

[3] T. A. Burton. Stability by fixed point theory for functional differential equations. Courier Corporation, 2013.

[4] C. Corduneanu. Integral equations and applications. Cambridge University Press, 1991.

[5] S. Czerwik. Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis, 1:5-11, 1998.

[6] K. Deimling. Nonlinear functional analysis. Springer Science & Business Media, 2013.

[7] A. A. B. Fayed, H. A. Makhzoum, R. A. Elmansouri, and A. K. Alshaikhly. Periodic solutions by Krasnoselskii fixed point theorem of neutral nonlinear system of dynamical equations with variable delays. International Journal of Scientific Research in Mathematical and Statistical Sciences, 8(1):1-10, 2021.

[8] A. A. B. Fayed, M. Illafe, H. A. Makhzoum, and R. A. Elmansouri. The existence and uniqueness of periodic solutions for nonlinear neutral dynamic equations with infinite delay on a time scale.

International Journal of Scientific Research and Management, 9(10):389-396, 2021.
[9] J. K. Hale. Theory of Functional Differential Equations. Springer-Verlag, New York, 1977.
[10] O. Diekmann, S. A. van Gils, S. M. VerduynLunel, and H.-O. Walther. Delay Equations: Functional-, Complex-, and Nonlinear Analysis. Springer-Verlag, New York, 1995.
[11] M. Illafe, H. Makhzoum, and F. Yousef. Periodic solutions and stability analysis of nonlinear neutral delay equations with infinite delay via fixed-point theorems. International Journal of Differential Equations, December 2025.
[12] V. Kolmanovskii and A. Myshkis. Introduction to the theory and applications of functional differential equations. Springer Science & Business Media, 2013.
[13] H. Makhzoum, H. G. Atetalla, A. Saleh, H. F. Layyas, et al. Existence and uniqueness results for delay integral equations in b-metric spaces. International Journal of Environmental Sciences, September 2025. DOI: 10.64252/sbk2vn37
[14] D. R. Smart. Fixed point theorems. Cambridge University Press, 1980.