Afro-Asian Journal of Scientific Research (AAJSR)
المجلة الأفروآسيوية للبحث العلمي
E-ISSN: 2959-6505
Volume 4, Issue 2, 2026
Page No: 124-129
Website: https://aajsr.com/index.php/aajsr/index
SJIFactor 2024: 5.028 ISI 2025: 0.915 معامل التأثير العربي (AIF) 2025: 0.76

A Hybrid Analytical-Geometric Methodology for Autonomous Linear Dynamical Systems

Adam Ibrahim Almedhak[1*], Salem Alnayhoum[2]
[1]Department of General Mathematics, Higher Institute of Science and Technology, Tocra, Libya
[2]Department of General Mathematics, Bright Star University, Libya

منهجية هجينة تحليلية-هندسية للأنظمة الديناميكية الخطية المستقلة

ادم ابراهيم محمد[1*]، سالم النيهوم[2]
[1]قسم المواد الرياضية العامة، المعهد العالي للعلوم والتقنية، توكرة، ليبيا
[2]قسم الرياضيات العامة، جامعة النجم الساطع، البريقة، ليبيا

*Corresponding author: medhakadam@gmail.com

Received: February 17, 2026	Accepted: April 01, 2026	Published: April 12, 2026

Abstract:

This investigation advances an integrated analytical and qualitative examination of autonomous linear differential systems, which constitute a fundamental paradigm for modeling dynamical phenomena across engineering, physics, and applied mathematics. The primary goal is to establish a comprehensive characterization of solution behavior and stability through a unified mathematical framework. The proposed methodology synergistically combines the Laplace transform with the Gauss-Jordan elimination technique to derive explicit closed-form solutions for coupled linear systems. This hybrid analytical protocol not only simplifies the solution procedure but also improves computational efficiency when addressing systems of elevated complexity. Furthermore, the qualitative evolution of solutions undergoes rigorous scrutiny via eigenvalue and eigenvector analysis. The findings demonstrate that the spectral characteristics of the system matrix play a decisive role in determining stability properties and asymptotic system dynamics. Equilibrium configurations receive systematic classification into stable nodes, unstable nodes, and saddle points, substantiated by geometric interpretation through phase plane analysis. A comparative evaluation of solution methodologies-encompassing the Laplace transform technique, eigenvalue-based algebraic approaches, and computational approximation methods-is additionally presented. The outcomes highlight that while analytical approaches furnish exact solutions and profound theoretical insight, numerical techniques afford flexibility in managing complex and large-scale systems. The integration of these distinct paradigms yields a more robust and holistic framework for dynamical system analysis.

Keywords: Autonomous linear differential systems, Laplace transform, Gauss–Jordan elimination, eigenvalues and eigenvectors, system stability.

الملخص:

تقدّم هذه الدراسة معالجة متكاملة تجمع بين التحليل الكمي والدراسة النوعية للأنظمة التفاضلية الخطية الذاتية، والتي تمثّل نموذجًا أساسيًا لنمذجة الظواهر الديناميكية في مجالات الهندسة والفيزياء والرياضيات التطبيقية. يتمثل الهدف الرئيس في بناء توصيف شامل لسلوك الحلول واستقرارها ضمن إطار رياضي موحّد. تعتمد المنهجية المقترحة على تكامل منهجي بين تحويل لا بلاس وتقنية الحذف بطريقة جاوس–جوردان، بهدف اشتقاق حلول صريحة مغلقة للأنظمة الخطية المقترنة. ولا يقتصر هذا النهج التحليلي الهجين على تبسيط إجراءات الحل فحسب، بل يسهم أيضًا في تحسين الكفاءة الحسابية عند

التعامل مع الأنظمة عالية التعقيد. علاوة على ذلك، يخضع التطور النوعي للحلول لتحليل دقيق من خلال دراسة القيم الذاتية والمتجهات الذاتية. وتُظهر النتائج أن الخصائص الطيفية لمصفوفة النظام تؤدي دورًا حاسمًا في تحديد خصائص الاستقرار والسلوك التقريبي للنظام. كما يتم تصنيف حالات التوازن بصورة منهجية إلى عقد مستقرة، وعقد غير مستقرة، ونقاط سرجيه، مع دعم هذا التصنيف بتفسير هندسي عبر تحليل مستوى الطور. وتتضمن الدراسة أيضًا تقييمًا مقارنًا لأساليب الحل المختلفة، بما في ذلك طريقة تحويل لا بلاس، والمقاربات الجبرية المعتمدة على القيم الذاتية، وطرائق التقريب الحاسوبية. وتبرز النتائج أن الأساليب التحليلية توفّر حلولًا دقيقة ورؤى نظرية عميقة، في حين تمنح الأساليب العددية مرونة أكبر في التعامل مع الأنظمة المعقدة وواسعة النطاق. ويؤدي دمج هذه المناهج المختلفة إلى بناء إطار أكثر قوة وشمولية لتحليل الأنظمة الديناميكية.

الكلمات المفتاحية: الأنظمة التفاضلية الخطية الذاتية، تحويل لا بلاس، حذف جاوس–جوردان، القيم الذاتية والمتجهات الذاتية، استقرار الأنظمة.

Introduction:

The study of autonomous linear differential systems admits multiple analytical and computational avenues of investigation. Among the most prominent are the Laplace transform technique, the eigenvalue decomposition method, and computational approximation algorithms. Each methodological pathway provides a distinct perspective regarding solution construction, stability interpretation, and computational economy. These systems constitute a fundamental class of dynamical models widely utilized in engineering analysis, physical theory, and applied mathematics. Their defining characteristic is the absence of explicit temporal dependence on the right-hand side of the governing equations. The canonical representation appears as $\frac{dx}{dt} = \mathbf{Ax}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ denotes a constant coefficient matrix. Prior work has extensively explored solution techniques for such systems, as documented in the literature [2,3,4]. The matrix exponential approach, eigenvalue-based expansions, and Laplace domain transformations each offer distinct advantages depending on the problem context [9,10]. Autonomous linear differential systems represent an indispensable instrument for modeling dynamical phenomena, as scholarly attention extends beyond mere solution acquisition toward understanding qualitative evolution and stability characteristics. The research problem emerges from the necessity to integrate analytical methods with geometric interpretation to achieve a more comprehensive grasp of these systems. The adopted methodology synthesizes the Laplace transform, Gaussian elimination, and phase plane analysis techniques. The investigation aims to illuminate the role of eigenvalues in determining system behavior and to establish a unified framework that links algebraic solutions with geometric visualization, grounded in a theoretical foundation drawn from differential equations and dynamical systems theory. The unique contribution of this work relative to prior studies lies in the systematic combination of the Laplace transform and Gauss-Jordan elimination within a single analytical protocol, coupled with explicit phase plane validation. While previous treatments have often treated these methods separately or in sequence without deep integration [1,5,8], the present framework demonstrates how their synergistic application enhances both computational clarity and theoretical insight. Furthermore, the explicit demonstration of the relationship between spectral properties and geometric trajectories under zero-initial-condition degenerate cases provides a pedagogical and analytical refinement not commonly emphasized in standard expositions [6,7].

Preliminaries:

Definition 3.1. A system of first-order differential equations is designated as autonomous when it can be expressed **in the form:**

$$\begin{aligned} \frac{dx_1}{dt} &= g_1(x_1, x_2, \dots, x_n) \\ \frac{dx_2}{dt} &= g_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(x_1, x_2, \dots, x_n) \end{aligned}$$

In such a formulation, the independent variable t does not appear explicitly on the righthand side of any constituent differential equation.

The general representation of an autonomous linear system may be written compactly **as:**

$$\mathbf{X}'(t) = \mathbf{AX}(t) \qquad (1)$$

where $\mathbf{A}$ is an $n \times n$ constant matrix.

Definition 3.2. The Laplace transform constitutes an integral transformation employed to facilitate the resolution of differential equations. For a function $y(t)$ defined for $t \geq 0$, the Laplace transform **is defined as:**

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt \qquad (2)$$

This transformation converts time-domain equations into frequency-domain algebraic relations expressed in terms of the complex variable s.

Theorem 3.1. Suppose that f is continuous and its derivative f' is piecewise continuous on any interval $0 \leq t \leq A$. Assume further that there exist constants K, a, and M such that $|f(t)| \leq Ke^{at}$ for $s > a$. Then the Laplace transform of the derivative satisfies $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.

Definition 3.3. The inverse Laplace transform is a mathematical operation that maps a function $F(s)$ in the complex frequency domain back to a function $f(t)$ in the time domain. **It is denoted as:**

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \qquad (3)$$

If $F(s)$ decomposes as $F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$ and each $f_i(t) = \mathcal{L}^{-1}\{F_i(s)\}$, then linearity implies $f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$.

The Gauss-Jordan elimination method provides a systematic technique for solving systems of linear equations. It proceeds by transforming the augmented matrix into reduced row echelon form through elementary row operations.

The general solution of the system in (1) is given by:

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}_0 \qquad (4)$$

This solution is also termed a path or trajectory, as the components $x_1(t), x_2(t), \ldots, x_n(t)$ parametrically define a curve. A constant solution $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}_0$ is designated a critical point or equilibrium solution.

Theorem 3.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct real eigenvalues of the coefficient matrix $\mathbf{A}$ in the homogeneous system, and let $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n$ denote the corresponding eigenvectors. Then the general solution on the interval $(-\infty, \infty)$ takes the form:

$$\mathbf{x} = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{k}_n e^{\lambda_n t} \qquad (5)$$

For the case of real distinct eigenvalues, three principal scenarios arise. When both eigenvalues are negative, the equilibrium constitutes a stable node. Solutions decay exponentially toward the origin, with the decay rate governed by the less negative eigenvalue for large times. When both eigenvalues are positive, an unstable node emerges, characterized by exponential divergence from equilibrium. When eigenvalues possess opposite signs, the critical point is classified as a saddle point, exhibiting attraction along one eigen-direction and repulsion along the other. This classification forms the cornerstone of linear stability analysis.

Main Results:

The analysis of autonomous linear differential systems fundamentally rests on converting the original time-domain problem into an algebraic framework that facilitates solution construction and interpretation. The prototypical system $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$ admits analytical solution via the Laplace transform, which transmutes differential equations into algebraic equations within the complex domain. This transformation markedly simplifies computational procedures, particularly for systems characterized by constant coefficients [1,4].

Applying the Laplace transform to both sides of the state equation yields:

$$s\mathbf{X}(s) - \mathbf{X}(0) = \mathbf{A}\mathbf{X}(s) \qquad (6)$$

which may be rearranged into:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{X}(0) \qquad (7)$$

This algebraic representation permits the systematic application of Gaussian elimination to solve for $\mathbf{X}(s)$, rendering the combined method both computationally efficient and structurally illuminating [2,8].

The eigenvalues of the coefficient matrix constitute the primary determinants of system dynamics.

The characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \qquad (8)$$

produces eigenvalues that directly govern the qualitative evolution of the system. Specifically, the signs of the real parts of the eigenvalues define stability conditions: negative real parts guarantee asymptotic stability, positive real parts indicate instability, and mixed signs yield saddle-type behavior [3,6]. This classification aligns precisely with classical stability theory in dynamical systems [4,7].

Solutions assume the exponential superposition form:

$$\mathbf{X}(t) = \sum_i c_i\mathbf{v}_i e^{\lambda_i t} \qquad (9)$$

demonstrating that system trajectories are inherently exponential, mirroring the spectral decomposition of matrix $\mathbf{A}$ [3].

On the basis of eigenvalue analysis, autonomous systems admit categorization into three principal types. A stable node occurs when all eigenvalues possess negative real parts. The system trajectories decay exponentially toward equilibrium, with $\mathbf{X}(t) \to \mathbf{0}$ as $t \to \infty$. This behavior typifies dissipative physical systems [5]. An unstable node arises when all eigenvalues have positive real parts. Solutions diverge exponentially from equilibrium, reflecting systems with positive feedback or amplification [6,7]. A saddle point emerges when eigenvalues display opposite signs. One component grows while the other decays, yielding extreme sensitivity to initial conditions. This configuration is structurally unstable and frequently appears in the linearization of nonlinear systems [7].

The innovative contribution of this study lies in the systematic integration of two complementary techniques: the Laplace transform for analytical simplification and Gaussian elimination for algebraic resolution. This hybrid approach bypasses direct eigenvector computation in early stages, provides a systematic pathway to derive solutions, and enhances computational clarity. Such integration has proven effective in solving coupled systems efficiently [8].

Analytical solutions correspond directly to geometric trajectories in the phase plane. Straight-line attraction toward equilibrium characterizes the stable node. Diverging paths indicate the unstable node. Hyperbolic curves identify the saddle point. This correspondence confirms that algebraic results align with geometric intuition, thereby reinforcing the theoretical framework [3].

Example: Saddle Point Dynamics:

Consider the autonomous linear system:

$$\begin{cases} x_1'(t) = 2x_1(t) + 3x_2(t) \\ x_2'(t) = 2x_1(t) + x_2(t) \end{cases}, t \geq 0 \qquad (10)$$

subject to initial conditions $x_1(0) = 0, x_2(0) = 0$. In matrix notation, $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$ with $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

Applying the Laplace transform to each equation and utilizing zero initial conditions yields:

$$sX_1(s) = 2X_1(s) + 3X_2(s)$$

$$sX_2(s) = 2X_1(s) + X_2(s)$$

Rearranging produces the algebraic system:

$$(s-2)X_1(s) - 3X_2(s) = 0$$

$$-2X_1(s) + (s-1)X_2(s) = 0$$

In matrix form, $(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{0}$. The augmented matrix for Gauss-Jordan elimination is:

$$\begin{pmatrix} s-2 & -3 & 0 \\ -2 & s-1 & 0 \end{pmatrix} \qquad (11)$$

Normalizing the first row by $\frac{1}{s-2}$ and performing elimination yields the reduced row echelon form, from which $X_1(s) = 0$ and $X_2(s) = 0$ are deduced. The inverse Laplace transform consequently gives $x_1(t) = 0$ and $x_2(t) = 0$ for all $t \geq 0$. This trivial solution corresponds to the equilibrium point.

To assess the intrinsic system behavior, **the characteristic equation of $\mathbf{A}$ is evaluated:**

$$\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda - 4 = 0 \qquad (12)$$

Factorization yields $(\lambda - 4)(\lambda + 1) = 0$, so $\lambda_1 = 4$ and $\lambda_2 = -1$. The origin is therefore a saddle point. For $\lambda_1 = 4$, solving $(\mathbf{A} - 4\mathbf{I})\mathbf{k} = \mathbf{0}$ gives eigenvector $\mathbf{k}_1 = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$. For $\lambda_2 = -1, (\mathbf{A} + \mathbf{I})\mathbf{k} = \mathbf{0}$ gives $\mathbf{k}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. **The general solution becomes:**

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \qquad (13)$$

Under zero initial conditions, $c_1 = 0$ and $c_2 = 0$, forcing $\mathbf{X}(t) = \mathbf{0}$. Although the trivial solution is obtained, any infinitesimal perturbation from the origin activates the unstable e^{4t} mode, causing exponential divergence along the unstable eigenvector. Only initial conditions lying exactly on the stable manifold lead to decay toward the origin. Thus, the zero solution reflects a degenerate excitation scenario rather than the full dynamical richness of the system. The coexistence of stable and unstable modes reveals a saddle-type equilibrium, whose long-term behavior is exquisitely sensitive to initial conditions.

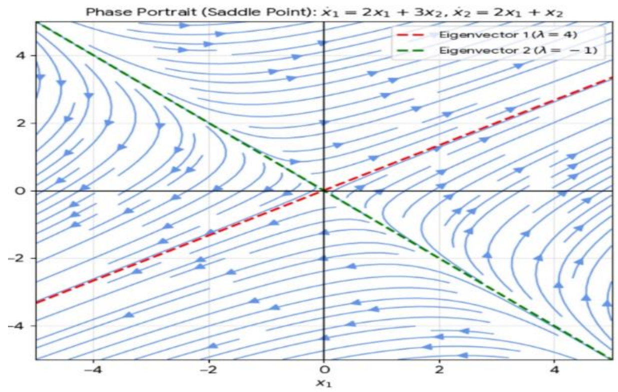

Figure (1): Phase plane representation of the saddle point dynamics. The red trajectory indicates the unstable manifold: any particle departing from this line moves indefinitely away from the origin. The green trajectory represents the stable manifold: it constitutes the unique set along which solutions approach the equilibrium. The origin $(0{,}0)$ constitutes an unstable equilibrium of the saddle type.

Example: Stable Node Dynamics:

Consider the autonomous linear system:

$$\begin{cases} x_1'(t) = -10x_1(t) + 6x_2(t) \\ x_2'(t) = 15x_1(t) - 19x_2(t) \end{cases} \qquad (14)$$

with zero initial conditions. **Applying the Laplace transform gives:**

$$sX_1(s) = -10X_1(s) + 6X_2(s)$$
$$sX_2(s) = 15X_1(s) - 19X_2(s)$$

Rearranged:

$$(s+10)X_1(s) - 6X_2(s) = 0$$
$$-15X_1(s) + (s+19)X_2(s) = 0$$

The augmented matrix is:

$$\begin{pmatrix} s+10 & -6 & 0 \\ -15 & s+19 & 0 \end{pmatrix} \qquad (15)$$

Gauss-Jordan elimination proceeds by normalizing the first row, eliminating below, and simplifying. The determinant condition for non-trivial solutions yields **the characteristic equation:**

$$s^2 + 29s + 100 = 0 \qquad (16)$$

Factorization gives $(s+25)(s+4) = 0$, so $s_1 = -25$ and $s_2 = -4$. For zero initial conditions, the Laplace-domain solution is $X_1(s) = 0, X_2(s) = 0$, implying $x_1(t) = 0$, $x_2(t) = 0$. The general solution ignoring the zero initial condition is:

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-25t} + c_2 \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-4t} \qquad (17)$$

Both eigenvalues are real and negative, hence the system is asymptotically stable. The e^{-25t} mode decays rapidly, while the e^{-4t} mode dominates long-term behavior. No oscillatory components appear due to the absence of imaginary parts.

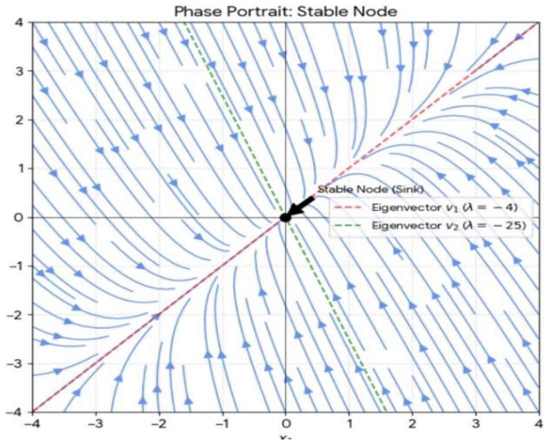

Figure (2): Phase plane portrait of the stable node system. All trajectories converge monotonically to the origin as $t \to \infty$, reflecting the asymptotic stability conferred by the two negative eigenvalues.

Conclusion:

This investigation has presented a comprehensive analytical and qualitative examination of autonomous linear differential systems. By synthesizing the Laplace transform with Gaussian elimination, explicit closed-form solutions were derived and systematically scrutinized. The results confirm that the stability characteristics and asymptotic evolution of these systems are entirely governed by the eigenvalues of the coefficient matrix. The hybrid framework not only simplifies the solution procedure but also enhances computational clarity, particularly for coupled systems of higher dimensionality. The consistency between analytical solutions and phase plane representations robustly demonstrates the synergy between algebraic manipulation and geometric visualization. The study further establishes that the Laplace-Gaussian framework is both computationally efficient and theoretically sound, a dual capability that enhances its applicability to real-world problems modeled by linear dynamical systems. The analysis of specific examples, including saddle point and stable node configurations, illustrates how the spectral properties dictate trajectory behavior even in degenerate zero-initial-condition cases. The geometric interpretation via phase plane analysis confirms that algebraic results correspond precisely to well-known trajectory classes: hyperbolic curves for saddle points, converging trajectories for stable nodes, and diverging paths for unstable nodes. This correspondence reinforces the theoretical foundations of linear dynamical systems. Future research may extend this integrated framework to nonlinear systems, time-dependent nonautonomous systems, and large-scale numerical simulations, thereby providing deeper insights into complex dynamical behavior. The methodological synergy demonstrated here opens avenues for more robust and unified treatments of dynamical phenomena across scientific and engineering disciplines.

References:

1. Boylestad, R. (2020). Introductory Circuit Analysis (13th ed.). Pearson.
2. Strang, G. (2016). Introduction to Linear Algebra (5th ed.). Wellesley-Cambridge Press.
3. Perko, L. (2001). Differential Equations and Dynamical Systems (3rd ed.). Springer.
4. Kreyszig, E. (2011). Advanced Engineering Mathematics (10th ed.). Wiley.
5. Ogata, K. (2010). Modern Control Engineering (5th ed.). Prentice Hall.
6. Hirsch, M. W., Smale, S., & Devaney, R. (2013). Differential Equations, Dynamical Systems (3rd ed.). Academic Press.
7. Strogatz, S. H. (2018). Nonlinear Dynamics and Chaos (2nd ed.). Westview Press.
8. Zill, D. G. (2018). Differential Equations with Boundary-Value Problems (9th ed.). Cengage Learning.
9. Edwards, C. H., & Penney, D. E. (2014). Differential Equations and Boundary Value Problems (5th ed.). Pearson.
10. Boyce, W. E., & DiPrima, R. (2017). Elementary Differential Equations (11th ed.). Wiley.