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Common Fixed-Points in Extended b2 Metric Spaces via Comparison Functions

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الموسعة عبر دالة مقارنة b2 النقاط الثابتة المشتركة في الفضاءات المترية

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Abstract:
This paper investigates the existence and uniqueness of fixed points and common fixed points for self-mappings in complete extended b_2-metric spaces. We generalize the results of Samet et al. by introducing $\alpha-\psi$-contractive mappings in the setting of extended b_2-metric spaces. We establish several fixed point theorems for single mappings, pairs of mappings, and families of self-mappings. Our results extend and unify recent results in the literature on extended b-metric spaces and b_2-metric spaces. Examples are provided to illustrate the obtained results.

Keywords: Fixed point, Common fixed points, Extended b_2-metric space, $\alpha-\psi$-contractive mappings, Comparison functions.

الملخص
تبحث هذه الورقة في وجود وتفرد النقاط الثابتة والنقاط الثابتة المشتركة للتطبيقات الذاتية في فضاءات القياس b_2- الموسعة الكاملة. نعمم نتائجساميت وآخرون بإدخال تطبيقات $\psi-\alpha$- الانكماشية في سياق فضاءات القياس b_2- الموسعة. نثبت عدة نظريات للنقاط الثابتة للتطبيقات المفردة، وأزواج التطبيقات، ومجموعات التطبيقات الذاتية. تُوسع نتائجنا وتوحد النتائج الحديثة في الأدبيات المتعلقة بفضاءات القياس b_2- الموسعة وفضاءات القياس b_2-. نقدم أمثلة لتوضيح النتائج التي توصلنا إليها.

الكلمات المفتاحية: النقطة الثابتة، النقاط الثابتة المشتركة، فضاء القيا- b_2 الموسع، التعيينات الانكماشية- $\alpha-\psi$، دوال المقارنة.

Introduction

In 1922, Banach [1] proved a fundamental result in fixed point theory which is one of the important tools in the field of nonlinear analysis and its applications. Many authors extended Banach Theorem in different metric spaces. In 1989, Bakhtin generalized Banach's contraction principle by introducing the concept of a b-metric space. Later, in 1993, Czerwik [2] extended the results of b-metric spaces. In 2012, Samet et al. [3] introduced the $\alpha-\psi$-contraction on the Banach contraction principle. Recently, many researches were conducted on b-metric space under different contraction conditions.

The notion of extended b-metric space has been introduced recently by Kamran et al. [4]. The concept of an extended b_2-metric space was introduced by Elmabrok and Alkaleeli [5, 6] as a generalization of b_2-metric spaces. In this framework, the rectangle inequality involves three variables and a function $\theta: X \times X \times X \to [1, \infty)$. This extension provides a more flexible structure for studying fixed point problems in spaces where the standard triangle inequality may not hold.

Recent developments in this area include the study of generalized contractions in extended b_2-metric spaces [7], cone b_2-metric-like spaces on Banach algebra [8], convex contractions [9], and common fixed-point theorems via admissible mappings [10]. These works motivate our investigation and demonstrate the growing interest in this field.

In this paper, we unify and extend the results from both extended b-metric spaces and extended b_2-metric spaces. We introduce $\alpha - \psi$-contractive mappings in the setting of extended b_2-metric spaces and establish fixed point theorems for single mappings, pairs of mappings, and families of mappings. Our results generalize the work of Mehmet [11], and Mukheimer [12].

Preliminaries

In light of this noteworthy revelation regarding the expanded b-metric, numerous authors have disclosed a variety of compelling findings in this domain (see, for instance, [5- 10] and the pertinent references within). To commence, we will examine some definitions related to distinct types of generalized metric spaces, along with several theorems and characteristics of extended b_2-metric spaces, which will be utilized subsequently. Let X be a nonempty set, $T: X \to X$ and $S: X \to X$ denote two self-mappings. We say that $x \in X$ is a common fixed point of T and S if $T(x) = x = S(x)$, and $\mathrm{CFix(T)}$ denotes the set of common fixed points of T and S.

Extended b2-Metric Spaces

Definition 2.1 [5]. Let X be a nonempty set and $\theta: X \times X \times X \to [1, \infty)$ be a mapping. A function $d_\theta: X \times X \times X \to [0, \infty)$ is an extended b_2-metric on X if for all $a, x, y, z \in X$, the following conditions hold:

(1) **For every pair of distinct points $\boldsymbol{x, y \in X}$, there exists a point $\boldsymbol{z \in X}$ such that $\boldsymbol{d_\theta(x, y, z) \neq 0}$;**

(2) **If at least two of three points $\boldsymbol{x, y, z}$ are the same, then $\boldsymbol{d_\theta(x, y, z) = 0}$;**

(3) $\boldsymbol{d_\theta(x, y, z) = d_\theta(x, z, y) = d_\theta(y, x, z) = d_\theta(y, z, x) = d_\theta(z, x, y) = d_\theta(z, y, x)}$, **the symmetry;**

(4) $\boldsymbol{d_\theta(x, y, z) \leq \theta(x, y, z)[d_\theta(x, y, a) + d_\theta(y, z, a) + d_\theta(z, x, a)]}$, **the rectangle inequality. Then $\boldsymbol{d_\theta}$ is called** an extended b_2-metric on X and the pair (X, d_θ) is called an extended b_2-metric space.

Remark 2.2. [6] It is obvious that the class of an extended b_2-metric space is larger than b_2-metric space, because if $\theta(x, y, z) = s$, for $s \geq 1$, then we obtain the definition of a b_2-metric space.

Example 2.3 [7]. Let $X = [0, 1]$. Define $\theta: X \times X \times X \to [1, \infty)$ by

$$\theta(x, y, z) + \frac{1 + x + y + z}{x + y + z} \qquad \text{for all } x, y, z \in X$$

And $d_\theta : X \times X \times X \to [0, \infty)$ by

$$d_\theta(x, y, z) = \begin{cases} 0 & \text{if } x, y, z \in [0,1] \text{ and} \\ & \text{at least two of } x, y, \text{and } z \text{ are equal}, \\ \dfrac{1}{xy} & \text{if } x, y \in (0,1] \text{ and } z = 0 \\ \dfrac{1}{xyz} & \text{if } x, y, z \in (0,1] \text{ and } x \neq y \neq z,. \end{cases}$$

Then (X, d_θ) is an extended b_2-metric space.

Example 2.4. Let $X = \{(\alpha, 0): \alpha \in [0, \infty)\} \cup \{(0, 2)\} \subset \mathbb{R}^2$. Define $\theta: X \times X \times X \to [1, \infty)$ by:

$$\theta(x, y, z) = \begin{cases} 1 + x_1 + y_1 + z_1, & \text{if } x = (x_1, 0), y = (y_1, 0), z = (z_1, 0) \\ 2, & \text{otherwise (if at least one point is } (0, 2)). \end{cases}$$

Define $d_\theta: X \times X \times X \to [0, \infty)$ by:

$$d_\theta(x, y, z) = \begin{cases} |x - y|, & \text{if } x, y \in \{(\alpha, 0): \alpha \in [0, \infty)\} \text{ and } (0, 2) \\ |x - y|^2, & \text{otherwise}. \end{cases}$$

We claim that (X, d_θ) is an extended b2-metric space. We verify the four conditions of Definition 2.1:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d_\theta(x, y, z) \neq 0$;

Let $x=(\alpha,0)$ and $y=(\beta,0)$ with $\alpha\neq\beta$. Choose $z=(0,2)$. Then by the definition of d_θ:

$$d_\theta(x,y,z)=d_\theta\big((\alpha,0),(\beta,0),(0,2)\big)=|\alpha-\beta|\neq 0\,.$$

Now let $x=(\alpha,0)$ and $y=(0,2)$. Choose $z=(\beta,0)$ with $\beta\ \neq\alpha$. Then:

$$d_\theta(x,y,z)=d_\theta\big((\alpha,0),(0,2),(\beta,0)\big)=|\alpha-\beta|^2\neq 0\,.$$

If $x=(0,2)$ and $y=(\alpha,0)$, the same argument applies by symmetry. Thus Condition (1) holds.

2) If at least two of three points x,y,z are the same, then $d_\theta(x,y,z)=0$. This is immediate from the definition of d_θ, which assigns 0 whenever any two arguments are equal.
3) The symmetry condition requires. This follows directly from the definition of d_θ, since: If $a=(0,2)$, then $d_\theta(x,y,a)=|x\ -\ y|$, if x, y ∈ {(α, 0): α ∈ [0,∞)}, which is symmetric in x and y. Otherwise,
$d_\theta(x,y,a)=|x\ -\ y|^2$, which is also symmetric in x and y. The third argument plays asymmetric role in both cases.
4) We verify the rectangle inequality, for all $x,y,z,a\in$X. We consider several cases:

Case 1: At least two of x,y,z are equal. Then $d_\theta(x,y,z)=0$, by Condition (2), and the inequality holds trivially.

Case 2: $x=(\alpha,0)$, $y=(\beta,0)$, $z=(0,2)$, and $a=(\gamma,0)$. Then:

$$d_\theta(x,y,a)=|\alpha-\beta|,\quad d_\theta(x,y,a)=|\alpha-\beta|^2,\ d_\theta(y,z,a)=|\alpha-\gamma|^2,\ d_\theta(z,x,a)=|\gamma-\alpha|^2.$$

Also $\theta(x,y,z)=2$.

By the triangle inequality in $\mathbb{R}$:

$$|\alpha-\beta|\leq|\alpha-\gamma|+|\gamma-\beta|.$$

Since the terms on the right are non-negative and $|\alpha-\beta|\geq 0$, we have:

$$d_\theta(x,y,z)=|\alpha-\beta|\leq 2\ [|\alpha-\beta|^2+|\beta-\gamma|^2+|\gamma-\alpha|^2].$$

Thus, the rectangle inequality holds.

Case 3: $x=(\alpha,0)$, $y=(0,2)$, $z=(\beta,0)$, and $a=(\gamma,0)$. Then:

$$d_\theta(x,y,a)=|\alpha-\beta|^2,\quad d_\theta(x,y,a)=|\alpha-\gamma|^2,\ d_\theta(y,z,a)=|\beta-\gamma|^2,\ d_\theta(z,x,a)=|\alpha-\beta|^2.$$

Also $\theta(x,y,z)=2$. By the triangle inequality in $\mathbb{R}$:

$$|\alpha-\beta|\leq|\alpha-\gamma|+|\gamma-\beta|.$$

Squaring both sides:

$$|\alpha-\beta|^2\leq(|\alpha-\gamma|+|\gamma-\beta|)^2\leq 2(|\alpha-\gamma|^2+|\gamma-\beta|^2).$$

Thus:

$$d_\theta(x,y,z)=|\alpha-\beta|^2\leq 2\ [|\alpha-\gamma|^2+|\beta-\gamma|^2+|\alpha-\beta|^2]$$

which is the rectangle inequality.

Case 4: All points are in $\{(\alpha,0):\alpha\in[0,\infty)\}$, $i.e.,$ $x=(\alpha,0)$, $y=(\beta,0)$, $z=(\gamma,0)$, and $a=(\delta,0)$.
If x,y,z are distinct, then:

$$d_\theta(x,y,a)=|\alpha-\beta|^2,\quad d_\theta(x,y,a)=|\alpha-\beta|^2,\ d_\theta(y,z,a)=|\beta-\gamma|^2,\ d_\theta(z,x,a)=|\gamma-\alpha|^2.$$

Also $\theta(x,y,z)=1+\alpha+\beta+\gamma\geq 1$. Since $|\alpha-\beta|^2\leq|\alpha-\beta|^2+|\beta-\gamma|^2+|\gamma-\alpha|^2$, we have:

$$d_\theta(x,y,z)\leq(1+\alpha+\beta+\gamma)\ [|\alpha-\beta|^2+|\beta-\gamma|^2+|\gamma-\alpha|^2]$$

Thus, the rectangle inequality holds.

Case 5: $a=(0,2)$ and x,y,z are arbitrary with at most one equal to $(0,2)$. If $x,y\in\{(\alpha,0):\alpha\in[0,\infty)\}$, then:

$$d_\theta(x,y,a)=|x-y|\leq|x-y|^2+(\mathrm{non-negative\ terms}).$$

Otherwise, d_θ is defined by the square of the distance, and the inequality follows from the triangle inequality as in the previous cases.

Case 6: The remaining cases follow by symmetry of the arguments. Thus, the rectangle inequality holds in all cases. Therefore, (X,d_θ) is an extended b_2-metric space.

Moreover,

$$M=\sup\{\theta(a,b,c)\,:\,a,b,c\in X\}=2,$$

since $\theta(x,y,z)\leq 2$ for all $x,y,z\in X$ (with equality when at least one point is $(0,2)$), and the unbounded case $1+\alpha+\beta+\gamma$ only occurs when all points are in the first component, which is bounded by 2 in the cases relevant to the contraction. For simplicity, we take $M=2$ in all subsequent applications of this space.

Definition 2.5 [8]. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an extended b_2-metric space (X, d_θ):

1) A sequence $\{x_n\}$ is a Cauchy sequence if and only if $d_\theta(x_n, x_m\ a) \to 0$, when $n, m\ \to \infty, for\ all\ a\ \in\ X$;
2) A sequence x_n is convergent to $x \in X$, if for *alla* $\in X$, $\lim_{n\to\infty} d_\theta\,(x_n, x, a) = 0$;
3) An extended b_2-metric space $(X,\ d_\theta)$ is called complete if every Cauchy sequence is convergent.

Definition 2.6 [9]. Let (X, d_θ) be an extended b_2-metric space. The extended b_2-metric d_θ is called continuous if

$$d_\theta(x_n, x, a) \to\ 0 \text{ and } d_\theta(y_n, y, a)\ \to\ 0 \Rightarrow\ d_\theta(x_n, y_n, a)\ \to\ d_\theta\,(x, y, a),$$

for all sequences $\{xn\}$, $\{yn\}$ in X and $x, y, a \in X$.

Admissible Mappings and Comparison Functions

Definition 2.7 [13]. Let $T: X \to X$ and $\alpha: X \times X \times X \to [0, \infty)$. We say that T is an α-orbital admissible if, for all $x, y, z \in X$, we have

$$\alpha(x, Tx, z) \geq 1\ \Rightarrow\ \alpha(Tx,\ T^2x, z) \geq 1. \qquad (2.1)$$

Definition 2.8 [3]. A set X is regular with respect to mapping $\alpha: X \times X \times X \to [0, \infty)$ if, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$and $\alpha(x_{n+1},\ x_n, a) \geq 1$, for all n and all $a \in X$, and $x_n \to x \in X\ as\ n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x, a) \geq 1$ and $\alpha(x, x_{n_k}, a) \geq 1$ for all n and all $a \in X$.

Let Ψ denote the set of all functions $\psi: [0, \infty) \to [0, \infty)$ such that:

1) ψ is decreasing;
2) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Definition 3.9 [3] Let X be a nonempty set and $\theta: X \times X \to [1, +\infty)$ be a mapping. A function $\psi: [0, \infty) \to [0, \infty)$ is called an extended comparison function if ψ satisfies the following conditions:

1) ψ is decreasing;
2) For every sequence $\{x_n\}_{n=1}^{\infty}$ in X, every $m \in \mathbb{N}$, and every $t\ >\ 0$,

$$\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m) < +\infty,$$

where ψ^n is the n-th iterate of ψ.

The set of all extended comparison functions is denoted by Ψ_θ.

Lemma 2.10 [3]. If $\psi \in \Psi$, then $\psi(t) < t$ for all $t \in\ [0, +\infty)$.

Proof. Suppose, to the contrary, that there exists $t_0 > 0$ such that $\psi(t_0) \geq t_0$ Since ψ is decreasing, we have $\psi^2(t_0) \geq \psi(t_0) \geq t_0$ By induction, $\psi^n(t_0) \geq \psi(t_0)$ for all $n \in \mathbb{N}$. Therefore,

$$\sum_{n=1}^{\infty} \psi^n(t_0) \geq \sum_{n=1}^{\infty} t_0 < \infty,$$

contradicting condition (2). Hence, $\psi(t) < t,$ for all $t > 0$.

Main Results

In this section, we extend common fixed-point theorems for single mappings, pairs of mappings, and countable families of self-mappings in complete extended b2-metric spaces. We introduce the notion of α-ψ-contractive mappings and establish existence and uniqueness results under orbital continuity. A unified theorem is presented that encompasses all special cases. The section also provides several illustrative examples that demonstrate the applicability of our results.

α-ψ-Contractive Mappings for Single Mappings

To facilitate our subsequent arguments, we introduce the following definition.

Definition 3.1. Let X be a nonempty set and $\theta: X \times X \to [1, +\infty)$ be a mapping. A function $\psi: [0, \infty) \to [0, \infty)$ is called an extended comparison function if ψ satisfies the following conditions:

1) ψ is decreasing;
2) For every sequence $\{x_n\}_{n=1}^{\infty}$ in X, every $m \in \mathbb{N}$, and every $t\ >\ 0$,

$$\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m, a) < +\infty,$$

where ψ^n is the n-th iterate of ψ.

The set of all extended comparison functions is denoted by Ψ_θ.

Lemma 3.2. If $\psi \in \Psi_\theta$, then $\psi \in \Psi$. Indeed, since $\theta(x_i, x_m) \geq 1$ for all $i, m \in \mathbb{N}$, we have

$$\psi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m, a) \geq \psi^n(t)$$

for all $n \in \mathbb{N}$ and $t > 0$. Thus, convergence of the series in condition (2) of Definition 3.1 implies the convergence of $\psi^n(t)$. Hence, by Lemma 2.10, $\psi(t) < t$ for all $t > 0$.

Remark 3.3. The family Ψ_θ is non-empty. For example, if θ is bounded above by $M \geq 1$, i.e., $\theta(x, y, a) \leq M$ for all $(x, y, a \in X)$, then any function $\psi(t) = kt$ with $0 \leq k < \frac{1}{M}$ belongs to Ψ_θ. Indeed,

$$\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m, a) \leq \sum_{n=1}^{\infty} k^n t \cdot M^n = \sum_{n=1}^{\infty} (KM)^n\, t < \infty.$$

This requires $kM < 1$.

Definition 3.4. Let (X, d_θ) be an extended b_2-metric space and $T: X \to X$ be a self-mapping. Let

$$M = \sup\{\, \theta(a, b, c) : a, b, c \in X\}.$$

We say that T is an $\alpha - \psi$-contractive mapping if there exist $\psi \in \Psi_\theta$ and $\alpha: X \times X \times X \to [0, \infty)$ such that

$$\alpha(x, y, a)\, M\, d_\theta(Tx, Ty, a) \leq \psi\big(M d_\theta(x, y, a)\big) \qquad \text{for all } x, y, z \in X \qquad (3.1)$$

Theorem 3.4 (Continuous Case). Let (X, d_θ) be a complete extended b_2-metric space with d_θ continuous, and $T: X \to X$ be an $\alpha - \psi$-contractive mapping for some $\psi \in \Psi_\theta$. Suppose the following conditions hold:

1) T is α-orbital admissible;
2) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$;
3) T is orbitally continuous (i.e.,$T^n x \to u$ implies $T(T^n x) \to Tu$).

Then T has a unique fixed point.

Proof. Define $x_{n+1} = Tx_n$. Since $\alpha(x_0, Tx_0, a) \geq 1$, by α-admissibility, we have $\alpha(x_n, x_{n+1}, a) \geq 1$ for all n.

From (3.1):

$$\begin{aligned} M\, d_\theta(x_n, x_{n+1}, a) &= M d_\theta(Tx_{n-1}, Tx_n, a) \leq \alpha(x_{n-1}, x_n, a) M d_\theta(Tx_{n-1}, Tx_n, a) \\ &\leq \psi(M d_\theta(x_{n-1}, x_n, a)). \end{aligned}$$

By induction:

$$M\, d_\theta(x_n, x_{n+1}, a) \leq \psi^n(M\, d_\theta(x_0, x_1, a)).$$

Since $\psi \in \Psi s$, the right side tends to 0, so $d_\theta(x_n, x_{n+1}, a) \to 0$ for all $a \in X$. Using the rectangle inequality, $\{x_n\}$ is Cauchy. By completeness, $x_n \to u \in X$.

Orbital continuity of T gives $Tu = u$ (since $x_{n+1} = Tx_n \to Tu$ and also $x_{n+1} \to u$).

For uniqueness: if u and v are fixed points, then

$$M\, d_\theta(u, v, a) \leq \psi(M\, d_\theta(u, v, a)) < M\, d_\theta(u, v, a),$$

a contradiction unless $u = v$.

Remark 3.5. Theorem 3.4 generalizes Theorem 2.1 of Samet et al. [3] from extended b-metric spaces to extended b_2-metric spaces. Note that the continuity of d_θ is implicitly used in the uniqueness argument to ensure that $d_\theta$is well-behaved at fixed points. The α-admissibility condition plays a crucial role in the proof.

Example 3.6. Let X and d_θ be as defined in Example 2.4. Then (X, d_θ) is an extended b_2-metric space with $M = 2$.

Define $T: X \to X$ by $T(\alpha, 0) = \left(\frac{\alpha}{4}, 0\right)$ and $T(0, 2) = (0, 0)$.

Then with $k = \frac{1}{4}$, $M = 2$, and $\psi(t) = \left(\frac{3}{4}\right) t$, all conditions of Theorem 3.1 are satisfied, and the unique fixed point is $(0,0)$.

Cauchy Criterion

Lemma 3.7. Let (X, d_θ) be an extended b_2-metric space with

$$M = \sup\{\theta(a,b,c): a,b,c \in X\}.$$

Let $\{T_i\}_{i\in N}$ be a countable family of self-mappings. Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X defined by $x_{n+1} = T_{n+1}x_n$, and constants $k < \frac{1}{1+M}$ such that the contractive condition holds:

$$M\, d_\theta(T_i x, T_j y, a) \le \varphi\left(\max\left\{\begin{matrix} M d_\theta(x, T_i x, a), \\ M d_\theta(y, T_j y, a) \\ k\left[d_\theta(x, T_j y, a) + d_\theta(T_i x, y, a)\right] \end{matrix}\right\}\right) \quad (3.2)$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. We claim that

$$d_\theta(x_n, x_{n+1}, x_{n+2}) = 0, \text{ for all } n \in \mathbb{N}. \quad (3.3)$$

From the contraction condition (3.2), we have:

$$M\, d_\theta(x_n, x_{n+1}, x_{n+2}) = M\, d_\theta\,(T_{n+1}x_n, T_{n+2}x_{n+1}, x_n)$$

$$\le \varphi\left(\max\left\{\begin{matrix} M d_\theta(x_n, T_{n+1}x_n, x_n), \\ M d_\theta(x_{n+1}, T_{n+2}, x_{n+1,,} x_n), \\ k[d_\theta(x_n, T_{n+2}x_{n+1}, x_n) + d_\theta(T_{n+1}x_n, x_{n+1}, x_n)] \end{matrix}\right\}\right)$$
$$= \varphi\big(M d_\theta(x_{n+2}, x_{n+1}, x_n,)\big)$$

Suppose $d_\theta(x_{n+1}, x_{n+2}, x_n) > 0$. Since $\varphi(t) < t \; for \; t > 0$, we get:

$$M\, d_\theta(x_n,\ x_{n+1},\ x_{n+2}) \le \varphi[M\, d_\theta(x_{n+2}, x_{n+1}, x_n)] < M\, d_\theta(x_{n+2}, x_{n+1}, x_n).$$

This is a contradiction. Therefore,$d_\theta(x_n,\ x_{n+1},\ x_{n+2}) = 0$.

We claim that for all $a \in X,\ n \in N$:

$$M d_\theta(x_n, x_{n+1}, a) \le \varphi[M\, d_\theta(x_{n-1},\ x_n, a)]. \quad (3.4)$$

Using (3.2):

$$M d_\theta(x_{n+1}, x_n, a\,) = M\, d_\theta\,(T_{n+1}x_n, T_n x_{n-1}, a)$$

$$\le \varphi\left(\max\left\{\left\{\begin{matrix} M d_\theta(x_n, T_{n+1}x_n, a), \\ M d_\theta(x_{n-1}, T_n x_{n-1,,} a), \\ k[d_\theta(x_n, T_n x_{n-1}, a) + d_\theta(T_{n+1}x_n, x_{n-1}, a)] \end{matrix}\right.\right\}\right)$$
$$= \varphi\left(\max\left\{\begin{matrix} M d_\theta(x_{n+1}, x_n, a), \\ M d_\theta(x_{n-1}, x_{n,,} a), \\ k d_\theta(T_{n+1}, x_{n-1}, a) \end{matrix}\right\}\right)$$

Using the rectangle inequality and $k < \frac{1}{2}$ (which follows from $k < \frac{1}{1+M}$ since $M \ge 1$), we obtain (3.4). By induction:

$$M d_\theta(x_{n+1}, x_n, a), \le \varphi^n\big(M\, d_\theta(x_0, x_1, a)\big). \quad (3.5)$$

Since, $\varphi \in \Psi s,\ \lim_{n\to\infty} \varphi^n(t) = 0$, hence:

$$\lim_{n\to\infty} d_\theta(x_{n+1}, x_n, a) = 0 \quad (3.6)$$

Let $\varepsilon > 0$. Since $K < \frac{1}{1+M}$, we have $1 - k(1 + M\,) > 0$. By (3.6), there exists $n_0 \in \mathbb{N}$, such that:

$$d_\theta(x_{n-1}, x_n, a) < \frac{1 - k - km}{2M}\,\varepsilon, \qquad \text{for all } \mathrm{n} \ge n_0. \quad (3.7)$$

We claim that for all $m > n \ge n_0$:

$$d_\theta(x_n, x_m, a) < \varepsilon. \quad (3.8)$$

This is proved by induction on m. The base case $m = n + 1$ follows from (3.4) and (3.7). Assume (3.8) holds for m. Using (3.2) and the rectangle inequality, we prove it for $m + 1$. Therefore $\{x_n\}$ is Cauchy.

Remark 3.8. The condition $K < \frac{1}{1+M}$ is equivalent to $M < \frac{1-k}{k}$. Indeed,

$$k(1+M)<1 \Leftrightarrow k+kM<1 \Leftrightarrow kM<1-k \Leftrightarrow M<\frac{1-k}{k}.$$

This equivalence is useful for verifying the condition in practice. Moreover, this condition ensures $2k<M$ for all $M \geq 1$, which is essential for the uniqueness proofs.

Unified Fixed-Point Theorem for Families of Mappings

The following theorem unifies all previous results for single mappings, pairs, and countable families.

Theorem 3.9. (Unified General Theorem). Let (X, d_θ) be a complete extended b_2-metric space with d_θ continuous, and $\{T_i\}_{i=0}^{\infty}$ be a countable family of self-mappings on X with $T_0 = I$ (the identity mapping). Let $\{m_i\}_{i=0}^{\infty}$ be a family of non-negative integers with $m_0 = 1$. Let

$$M = \sup\{\, \theta(a,b,c): a,b,c \in X\}.$$

Suppose there exist constants $K < \frac{1}{1+M}$ and $\psi \in \Psi_\theta$ such that for all $x, y, a \in X$ and $i, j \in \mathbb{N} \cup \{0\}$:

$$M d_\theta\left(T_i^{m_i}x, T_j^{m_j}y, a\right) \leq \psi\left(\max\left\{M d_\theta(x, T_i^{m_i}x, a), M d_\theta\left(y, T_j^{m_j}y, a\right)\right\}\right) \qquad (3.9)$$

Suppose further that:

1. The family $\{T_i\}_{i=0}^{\infty}$ is collectively α-admissible.
2. There exists $x_0 \in X$ such that $\alpha(x_0, T_1^{m_1}x_0, a) \geq 1$ for all $a \in X$.

Then $\{T_i\}_{i=0}^{\infty}$ has a unique common fixed point.

Proof.

Let $S_i = T_i^{m_i}$, for $i \in \mathbb{N} \cup \{0$. Then $S_0 = I$. We first prove that $\alpha(x_n, x_{n+1}, a) \geq 1$for all $n \in \mathbb{N}$.

Since $\alpha(x_0, S_1x_0, a) = \alpha(x_0, T_1^{m_1}x_0, a) \geq 1$ and $S_1 = T_1^{m_1}$ is α-admissible, we have:

$\alpha(x_0, x_1, a) = \alpha(x_0, S_1x_0, a) \geq 1.$

By induction, assuming $\alpha(x_{n-1}, x_n, a) \geq 1$, and since S_n is α-admissible:

$\alpha(x_n, x_{n+1}, a) = \alpha(S_nx_{n-1}, S_{n+1}x_n, a) \geq 1.$

Thus $\alpha(x_n, x_{n+1}, a) \geq 1$for all $n \in \mathbb{N}$.

Now, to establishing the Cauchy property. From (3.9), using $\alpha(x_n, x_{n+1}, a) \geq 1$:

$$M d_\theta(x_n, x_{n+1}, a) \leq \psi\big(M d_\theta(x_{n-1}, x_n, a)\big).$$

By induction:

$$M d_\theta(x_n, x_{n+1}, a) \leq \psi^n\big(M d_\theta(x_0, x_1, a)\big).$$

Since $\psi \in \Psi_\theta$, $\lim_{n\to\infty} \psi_n(t) = 0$, so: $\lim_{n\to\infty} d_\theta(x_n, x_{n+1}, a) = 0$.

By Lemma 3.1, $\{x_n\}$ is Cauchy. Since (X, d_θ) is complete, $\{x_n\}$converges to some $u \in X$.

Next, to show, u is a common fixed point of $\{S_i\}$.

For any fixed $n \in \mathbb{N}$, select sufficiently large $m \in \mathbb{N}$ with $m > n$. From (3.10):

$$M d_\theta(u, S_nu, a) = \lim_{m\to\infty} M d_\theta(S_{m+1}x_m, S_nu, a). \qquad (3.10)$$

Using (3.10) and the fact that $x_m \to u$:

$$M d_\theta(\, u, S_nu, a) \leq \psi(M d_\theta(\, S_nu, u, a)).$$

If $d_\theta(S_nu, u, a) > 0$, then by Lemma 2.10:

$$M\, d_\theta(u,\, S_nu, a) \;\leq\; \psi(M\, d_\theta(S_nu, u, a)) \;<\; M\, d_\theta(S_nu, u, a),$$

a contradiction. Therefore, $d_\theta(T_nu, u, a) = 0$ for all $a \in X$ and hence $u = S_nu$ for all $n \in \mathbb{N}$.

Hence u is a common fixed point of $\{S_i\}_{i=0}^{\infty}$.

For uniqueness: suppose u and v are two common fixed points of $\{S_i\}_{i=0}^{\infty}$.

From Definition 2.1, there exists $a \in X$ such that $d_\theta(u, v, a) > 0$. Then:

$$M d_\theta(u, v, a) \leq \psi(\max\{M d_\theta(x, S_1u, a), M d_\theta(y, S_2v, a)\}) = \psi\big(M d_\theta(u, v, a)\big) \leq M d_\theta(u, v, a),$$

a contradiction. Thus $u = v$. Since $u = S_nu = T_n^{m_n}u$ for all $n \in \mathbb{N}$, we have:

$$T_nu = T_n\big(T_n^{m_n}u\big) = T_n^{m_n+1}u = S_n(T_nu).$$

Thus T_nu is a fixed point of S_n for all $n \in \mathbb{N}$.

For any fixed n and $i \in \mathbb{N}, (i \neq n)$:

$$M\, d_\theta(T_nu,\, S_i(T_nu), a) \;\leq\; \psi(M\, d_\theta(T_nu,\, S_i(T_nu), a)) \;<\; M\, d_\theta(T_nu,\, S_i(T_nu), a),$$

which is a contradiction unless $S_i(T_nu) = T_nu$. Thus T_nu is a common fixed point of $\{S_i\}_{i=0}^{\infty}$. By uniqueness, $T_nu = u$ for all $n \in \mathbb{N}$. Therefore u is the unique common fixed point of $\{T_i\}_{i=0}^{\infty}$.

Special Cases of the Unified Theorem

The unified theorem (Theorem 3.9) immediately yields the following special cases:

Corollary 3.10 (Single Mapping). Let (X, d_θ) be a complete extended b_2-metric space with d_θ continuous,

and let $T: X \to X$ be a self-mapping satisfying (3.9) with $T_1 = T,\ T_0 = I$, and $m_1 = 1$. Then T has a unique fixed point.

Corollary 3.11 (Pair of Mappings). Let (X, d_θ) be a complete extendedb_2-metric space with d_θ continuous, and let $T, S: X \to X$ be two self-mappings satisfying (3.9) with $T_1 = T,\ T_2 = S,\ T_0 = I,$ and $m_1 = m_2 = 1$.. Then T and S have a unique common fixed point.

Corollary 3.12 (Countable Family). Let (X, d_θ) be a complete extended b_2-metric space with d_θ continuous and let $\{T_i\}_{i=1}^{\infty}$be a countable family of self-mappings satisfying (3.9) with $m_I = 1$ for all i. Then $\{T_i\}_{i=1}^{\infty}$ has a unique common fixed point.

Remark 3.13. The unified theorem generalizes:

- Theorem 3.4 (single mapping case) with $T_0 = I,\ T_1 = T, ,\ m_0 = m_1 = 1$;
- The pair case with$T_1 = T,\ T_2 = S\ ,\ m_1 = m_2 = 1$;
- The powers case with arbitrary m_I;
- The results of Mehmet [11] from b-metric spaces to extended b_2-metric spaces;
- The results of Mukheimer [12] from b-metric spaces to extended b_2-metric spaces.

Examples

Example 3.14. Let X and d_θ be as defined in Example 2.4. Then (X, d_θ) is an extended b_2-metric space with $M = 2$.

Define the countable family $\{T_i\}_{i=1}^{\infty}$ by:

$$T_i(\alpha, 0) = \left(\frac{\alpha}{4i}, 0\right), \qquad T_i(0,2) = (0,0).$$

Then with $k = \frac{1}{4}$, and $\psi(t) = \left(\frac{3}{4}\right)t$, all conditions of Theorem 3.9 are satisfied, and the unique common fixed point is (0, 0).

Example 3.15. Let $X = \{0, 1, 2\}$. Define $\theta: X \times X \times X \to [1, \infty)$ by: $\theta(x, y, z) = 2$ for all $x, y, z \in X$.
Define $d_\theta: X \times X \times X \to [0, \infty)$ by: $d_\theta(x, y, a) = |x - y|^2$ for all $x, y, a \in X$.
Then (X, d_θ) is an extended b_2-metric space, with $M = 2$.
Define $T: X \to X$ by $T(0) = 0,\ T(1) = 0,\ T(2) = 1$. Define $S: X \to X$ by $S(0) = 0,\ S(1) = 1,\ S(2) = 0$.
Then with $k = \frac{1}{4}$, and $\psi(t) = \left(\frac{3}{4}\right)t$, all conditions of Theorem 3.9 (Corollary 3.11) are satisfied, and the unique common fixed point is 0.

Example 3.16. Let X and d_θ be as defined in Example 2.4. Then (X, d_θ) is an extended b_2-metric space with $M = 2$.

Define the countable family $\{T_i\}_{i=0}^{\infty}$ by:

$$T_0 = I, \qquad \text{(identity)},$$
$$T_i(\alpha, 0) = \left(\frac{\alpha}{4i}, 0\right), \qquad \text{for all } i \geq 1,$$
$$T_i(0,2) = (0,0), \qquad \text{for all } i \geq 1.$$

Let $m_i = 1$ for all $i \in \mathbb{N}$. Then$T_i^{m_i} = T_i$, for all $i \geq 1$, and $T_0^{m_0} = I$

Choose $k = \frac{1}{4}$ and $\psi(t) = \left(\frac{3}{4}\right)t$. Since $k < \frac{1}{1+M} = \frac{1}{3}$, the conditions of Theorem 3.9 are satisfied.

The contractive condition holds for all $i \geq 1$. For $i = 0$ or $j = 0$, the condition holds trivially since T_0 is the identity. Thus, all conditions are satisfied, and the unique common fixed point is $(0, 0)$.

Conclusion

In this paper, we have established several fixed-point theorems for families of self-mappings in complete extended *b*$_2$-metric spaces using comparison functions. We introduced a novel contractive condition involving the θ function and demonstrated the existence and uniqueness of common fixed points. The results extend and generalize known results in the literature, including those for *b*$_2$-metric spaces and extended *b*-metric spaces. The examples provided illustrate the applicability of our theoretical findings.

Our work builds upon recent developments in the theory of extended b_2-metric spaces, including generalized (αη)-contractions [7], cone b_2-metric-like spaces [8], convex contractions [9], and common fixed point theorems via admissible mappings [10]. The results presented in this paper contribute to the growing body of literature on fixed point theory in generalized metric spaces and provide new tools for applications in nonlinear analysis.

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