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Complete Correspondence between the Quotient block topology and the Block-Cut Tree

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تناظر كامل بين تبولوجيا قسمة الكتل وشجرة الكتل والقطع

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Abstract:
The block topology $\tau_B(G)$, generated by blocks of a connected graph G, gives rise to a quotient space $Q(G)$ obtained by the identifying topologically indistinguishable vertices. In this paper, our main result, the Hasse-block-cut correspondence, shows that the Hasse diagram of specialization poset of $Q(G)$ is isomorphic to the block-cut tree $\mathcal{BC}(G)$, this holds under the condition that G has no internal bridge blocks (bridges between two cut vertices). This correspondence is a complete invariant as two such graphs have homeomorphic quotient spaces exactly when their block-cut trees are isomorphic, we also prove that any graph isomorphism induces a homeomorphism on the quotient spaces. In addition, we demonstrate that the degree of a node in $\mathcal{BC}(G)$ is equal to the number of Hasse neighbors of the corresponding point in $Q(G)$. out results establish $Q(G)$ can serve as a natural topological model for block decomposition, connecting graph theory and finite topology.

Keywords: Block topology, Alexandroff space, Quotient space, Hasse diagram, Block-cut tree.

الملخص:
تبولوجيا الكتل τ_G، المتولدة من كتل (blocks) رسم بياني مترابط G ، تُنتج فضاء قسمة $Q(G)$ يُبنى بتطابق الرؤوس الغير قابلة للتميز تبولوجياً (topologically indistinguishable vertices). في هذه الورقة، نثبت كنتيجة رئيسية تناظر هاس-الكتل والقطع (Hasse-block-cut correspondence) بين مخطط هاس (Hasse diagram) لمجموعة التخصيص الجزئية (specialization poset) للفضاء $Q(G)$ وشجرة الكتل والقطع $\mathcal{BC}(G)$، بشرط ألا تحتوي G على كتل الجسر الداخلية (جسورتصل بين نقطتي قطع). يُعد هذا التناظر ثابت تام (complete invariant): فرسمان بيانيان من هذا النوع لهما فضائي قسمة متشاكلين تبولوجيا (homeomorphic) إذا وفقط إذا كانت شجرتا الكتل والقطع الخاصتان بهما متشاكلتين بيانيا. كما نبين أن أي تشاكل بياني بين رسمين بيانيين يستحدث تشاكلا تبولوجيا بين فضاءات القسمة المقابلة لهما. بالإضافة إلى ذلك، نثبت أن درجة أي عقدة في الشجرة $\mathcal{BC}(G)$ تساوي عدد جوارات هاس (Hasse neighbors) للنقطة المقابلة لها في فضاء $Q(G)$. ترسي هذه النتائج $Q(G)$ كنموذج تبولوجي طبيعي لتجزئة الكتل، رابطة بذلك بين نظرية الرسم البياني (graph theory) والتبولوجيا المنتهية (finite topology).

الكلمات المفتاحية: تبولوجيا الكتل، فضاء أليكسندروف، فضاء القسمة، مخطط هاس، شجرة الكتل ونقاط القطع.

Introduction:

The construction of topological spaces on the vertex set of graphs G has recently garnered significant attention. Within this approach, topologies have been defined via adjacency and independence relations, and by other structural relations [1,2,3,4]. Among these is block topology $\tau_{\mathrm{B}}(\mathrm{G})$, which is generated from vertex sets of blocks in a graph G, was first introduced by Macaso and Balingit [5]. In our previous study [6], we examined the block space in depth: we characterized density, connectedness, contractibility and several other properties; we proved that it is a finite Alexandroff space [7] with B_v as minimal open neighborhood for each vertex v; and obtained a complete classification between the block spaces and their block-cut trees with equal corresponding blocks size. However, the block space is not generally T_0 , the standard resolution is to pass to the quotient topology $\mathcal{Q}(\mathrm{G}) = \left(\mathrm{V}/\!\sim_B, \tau_q(\mathrm{G})\right)$ as this restores the T_0 property, $\mathcal{Q}(\mathrm{G})$ is, based on the correspondence between Alexandroff T_0 −spaces and posets which formalized by stong [8] and developed by Barmak [9], is fully determined by its specialization poset. If so, the block-cut tree can be extracted directly from the topology of $\mathcal{Q}(\mathrm{G})$ with no loss of structural information, this connection, to the best knowledge, remains unexplored in the literature. In this paper, we resolve this question affirmatively: our main contribution is the Hasse-block-cut correspondence theorem (theorem (12)), which establishes an isomorphism between the Hasse diagram of the specialization poset of $\mathcal{Q}(\mathrm{G})$ and the block-cut tree $\mathcal{BC}(\mathrm{G})$, under the condition that the graph G has no internal bridge blocks. From this correspondence, $\mathcal{Q}(\mathrm{G})$ serves as a complete invariant of block-cut tree: each structure characterizes the other uniquely and also carrying the same the information. As a consequence, we derive a simple formula which determines the degree of any node of the block-cut tree directly from the number of Hasse neighbors of the corresponding point in $\mathcal{Q}(\mathrm{G})$. Moreover, we prove that every graph isomorphism induces a homeomorphism between the corresponding quotient spaces. All of these results uncover a precise dictionary between the topology of $\mathcal{Q}(\mathrm{G})$ and the combinatorics of the block-cut tree, as discussed in the main Section and demonstrated by the worked examples.

Preliminaries:

This section reviews the notation, terminology and basic concepts needed in the sequel unless otherwise mentioned. For additional details on graph theory, see [10,11,12,13]; for background on order theory and finite topology, see [8,9,14]

- **Definition (1):** A simple undirected graph G is an order pair $\left(\mathrm{V(G)}, \mathrm{E(G)}\right)$, where $\mathrm{V(G)}$ is a non-empty set of vertices, and $\mathrm{E(G)}$ is a set of edges linking any pair of vertices of G. A graph H is a subgraph of G if $\mathrm{V(H)} \subseteq \mathrm{V(G)}$ and $\mathrm{E(H)} \subseteq \mathrm{E(G)}$. A graph G is connected if there is a path connecting every pair of its vertices. A component of G is a maximal connected subgraph. A vertex $v \in \mathrm{V(G)}$ is a cut-vertex if removing v increases the number of components of G. A block of G is a maximal connected subgraph without cut-vertices. The simplest possible block in G is an isolated vertex. Two distinct blocks share at most one vertex, that vertex is necessarily a cut-vertex. The bridge block is internal if its two endpoints are cut-vertices. A block−cut tree $\mathcal{BC}(\mathrm{G})$ of a connected graph G is a bipartite graph whose vertices are the blocks an cut-vertices of G , with an edge between a cut-vertex node c_v and a block-node b_B if and only if $v \in \mathrm{V(B)}$. A pure vertex $v \in \mathrm{V(G)}$ is neither a cut-vertex nor isolated.

All graphs considered in this paper are simple and undirected, with block set $\mathcal{B}(\mathrm{G})$, and cut-vertex set $\mathcal{C}(\mathrm{G})$.

- **Definition (2):** A preorder on a set P is a binary relation that is reflexive, and transitive. A partial order is an antisymmetric preorder. A poset is a set equipped with a partial order. an element $b \in P$ covers the element $a \in P$ if $a \prec b$ and there is no other element c satisfies such that $a \prec c \prec b$. A Hasse diagram is a graphical representation of a finite poset $\mathcal{P}$, where each element of P is represented as a node, with an edge from a to b whenever b covers a, drawn upwards.
- **Definition (3):** A topology τ on a non-empty set X is a collection of subsets of X, which are called open, such that τ is closed under arbitrary unions and finite intersections, and contains empty set and X . The pair (X, τ) is called a topological space. If every intersection of open sets is open, then (X, τ) is called an Alexandroff space, this equivalent to every point $x \in \mathrm{X}$ has a minimal open neighborhood U_x. The specialization preorder of an Alexandroff space (X, τ) defined by:

$$x \preccurlyeq y \iff x \in \overline{\{y\}} \iff U_x \supseteq U_y$$

The collection $\mathfrak{B}$ of open subsets of X is a base for a topology τ if every non-empty open set in τ is the union of members of $\mathfrak{B}$. A collection $\mathcal{S}$ is a subbase for a topology τ if its finite intersections form a base for τ . For a surjective map $q\colon \mathrm{X} \to \mathrm{Y}$, The quotient topology on Y is $\tau_{\mathrm{Y}} = \{\mathrm{U} \subseteq$

$\mathrm{Y}: q^{-1}(\mathrm{U}) \in \tau_{\mathrm{X}}\}$. The quotient space of $\mathrm{X}/\sim$ is the set of equivalence classes $\{[x]: x \in \mathrm{X}\}$ with the quotient topology induced by the canonical projection $\pi: \mathrm{X} \to \mathrm{X}/\sim$ defined by $\pi(x) = [x]$, where $\sim$ is an equivalence relation on X.

Note that throughout this paper, our convention $(x \preccurlyeq y \Leftrightarrow U_x \supseteq U_y)$ which reverses the definition used in some sources like [9], where $(x \preccurlyeq y \Leftrightarrow U_x \subseteq U_y)$. This means, the open sets here correspond to up-sets rather than down-sets. In both cases, the two conventions are equivalent up to order reversal .

- **Definition (4):** [5,6] The block topology $\tau_{\mathrm{B}}(\mathrm{G})$ of the graph G on the vertex set $\mathrm{V}(\mathrm{G})$ is a topology that generated by the subbasis $\mathcal{S}_B = \{\mathrm{V}(B_1), \mathrm{V}(B_2), \dots, \mathrm{V}(B_k)\}$, with blocks $B_1, B_2, \dots, B_k$. The pair $(\mathrm{V}(\mathrm{G}), \tau_{\mathrm{B}}(\mathrm{G}))$ is called block topological space, simply block space. The minimal open neighborhood of a vertex v be denoted B_v, which is intersection of all blocks containing v.

The Hasse-Block-Cut Correspondence:

In [6], we proved that any block space $(\mathrm{V}, \tau_{\mathrm{B}}(\mathrm{G}))$ is an Alexandroff space with smallest open neighborhood B_v for each $v \in \mathrm{V}$. Consequently, the specialization relation of $(\mathrm{V}, \tau_{\mathrm{B}}(\mathrm{G}))$ is $u \preccurlyeq v \Leftrightarrow u \in \overline{\{v\}} \Leftrightarrow \mathrm{B}_u \supseteq \mathrm{B}_v$.

1. **Proposition:**

The specialization relation of $(\mathrm{V}, \tau_{\mathrm{B}}(\mathrm{G}))$ is preorder. Moreover, pure vertices are the minimal elements, and cut-vertices are maximal elements which are above the pure vertices in same block B , while isolated vertices are both minimal and maximal.

Proof. For reflexivity: for every $v \in \mathrm{V}(\mathrm{G})$ have $\mathrm{B}_v \supseteq \mathrm{B}_v$, then $v \preccurlyeq v$. For transitivity: if $u \preccurlyeq w \preccurlyeq v$, this means $\mathrm{B}_u \supseteq \mathrm{B}_w \supseteq \mathrm{B}_v$ and so $\mathrm{B}_u \supseteq \mathrm{B}_v$, then $u \preccurlyeq v$. In the second part : if v is a cut-vertex, $\mathrm{B}_v = \{v\}$ this means that there is no vertex u such that $v \prec u$, so cut-vertices are maximal among any other basis elements, and if $u \in \mathrm{V}(B)$ is a pure vertex, $\mathrm{B}_u = \mathrm{V}(B) \supseteq \{v\}: v$ is cut-vertex in B, then $\mathrm{B}_u \supseteq \mathrm{B}_v$. In particular, for each pure vertex w, $\mathrm{B}_w \supseteq \mathrm{B}_u$ implies that u, w lie in the same block and then $\mathrm{B}_w = \mathrm{B}_u$. Thus there is no element strictly between cut-vertices and pure vertices in the same block, and indeed the preorder has exactly height 1. Consequently, pure vertices of a block B are minimal elements directly below any cut-vertex within its block B. Finally, if v is isolated, then $\mathrm{B}_v = \{v\}$ and it is incomparable with any other vertex in $\mathrm{V}(B)$, and so there is no any elements lie below or above it. Thus v is both minimal and maximal. □

2. **Remark:** In general, the block topology $\tau_B(\mathrm{G})$ is not necessary T_0. As a result, the specialization preorder on $\mathrm{V}(\mathrm{G})$ may fail to be partial order, this occurs if and only if every block of G contains at most one pure vertex, more precisely, iff G is a forest (i.e., every block is K_1 or K_2).
3. **Definition:** For block space $(\mathrm{V}, \tau_{\mathrm{B}})$, define the equivalence relation $\sim_B$ by

$$u \sim_B v \iff \mathrm{B}_u = \mathrm{B}_v$$

Where the canonical map is $\pi_{\mathrm{G}}: \mathrm{V} \to \mathrm{V}/\sim_B$, and $(\mathrm{V}/\sim_B, \tau_q(\mathrm{G}))$ is the quotient space (simply $\mathcal{Q}(\mathrm{G})$).

4. **Proposition**: For each connected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, and $v \in \mathrm{V}$. The equivalence class $[v]$ of $\sim_B$ is

$$[v] = \begin{cases} [c] = \{c\} & ;\ v = c \text{ is a cut} - \text{vertex} \\ [B] = \mathrm{V}(B) \setminus \mathcal{C}(\mathrm{G}) & ;\ v = p \text{ is a pure vertex in block } B \end{cases}$$

Proof. The equivalence class $[v]$ of $\sim_B$ is $\{u \in \mathrm{V}: \mathrm{B}_u = \mathrm{B}_v\}$.

- **Case 1:** Let c is cut-vertex, so $\mathrm{B}_c = \{c\}$, therefore, $[c] = \{u: \mathrm{B}_u = \{c\}\} = \{c\}$.
- **Case 2:** Let p is pure vertex in the block B. Take $u \in \mathrm{V}(\mathrm{G})$, if $u \in \mathrm{B}_p$ and it is also pure vertex, then $p \in \mathrm{B}_u$ and $\mathrm{B}_u = \mathrm{B}_p$, thus $u \in [p]$, and if $u \in \mathrm{B}_p$ but it is cut-vertex, then $\mathrm{B}_u = \{u\}$, so $\mathrm{B}_u \subset \mathrm{B}_p$ while $\mathrm{B}_p \not\subseteq \mathrm{B}_u$, hence $u \notin [p]$, and if $u \notin \mathrm{B}_p$, then B_p and B_u are disjoint, so $\mathrm{B}_u \neq \mathrm{B}_p$.

From all last cases, the equivalent class of a pure vertex p is $\mathrm{V}(\mathrm{B}_p) \setminus \mathcal{C}(\mathrm{G})$. □

On the quotient set $\mathrm{V}/\sim_B$, we consider an order $\preccurlyeq_{\sim_B}$ directly derived from $\preccurlyeq$;

5. **Proposition:** The quotient set $(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B})$; $[u] \preccurlyeq_{\sim_B} [v] \Leftrightarrow \mathrm{B}_u \supseteq \mathrm{B}_v$ is partial order.

Proof. Reflexivity and transitivity of $\preccurlyeq_{\sim_B}$ hold directly from $\preccurlyeq$. For antisymmetry: Take $[u] \preccurlyeq_{\sim_B} [v]$ and $[v] \preccurlyeq_{\sim_B} [u]$, so $\mathrm{B}_u \supseteq \mathrm{B}_v$ and $\mathrm{B}_v \supseteq \mathrm{B}_u$, this means that $\mathrm{B}_u = \mathrm{B}_v \Leftrightarrow u \sim_B v \Leftrightarrow [u] = [v]$. Consequently, the quotient set is partial order. □

6. **Corollary:** The quotient poset $(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B})$ characterized by:

$$[B] \preccurlyeq_{\sim_B} [c] \iff c \in \mathrm{V}(B)$$

For each block B and cut-vertex c. In particular, the block classes $[B]$ of any vertex of B are minimal elements of $\mathrm{V}/{\sim_B}$, and cut-vertex classes $\{[c]: c \in \mathcal{C}(\mathrm{G})\}$ are maximal elements.

7. **Proposition:** Let G be a connected graph without internal bridge blocks. Then Hasse diagram of quotient poset $\left(\mathrm{V}/{\sim_B}, \preccurlyeq_{\sim_B}\right)$ is isomorphic to the block-cut tree $\mathcal{BC}(\mathrm{G})$ of G.

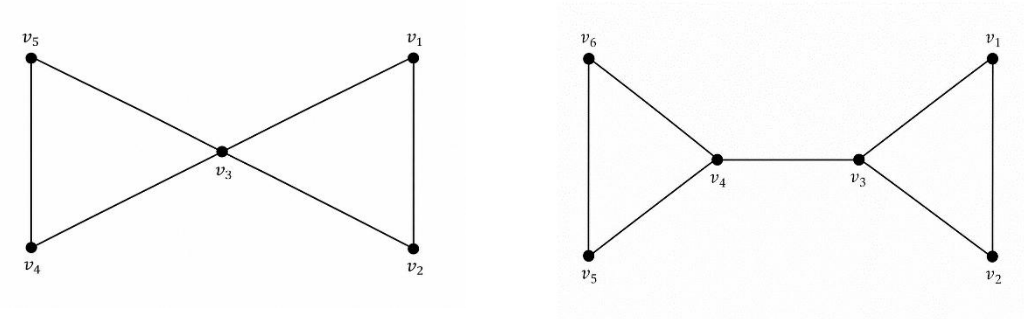
Proof. Define $\psi_2 : \mathrm{V}/{\sim_B} \longrightarrow \mathrm{V}(\,\mathcal{BC}(\mathrm{G})\,)$, by:

$$\psi_2([c]) = c_v \quad \forall\, c \in \mathcal{C}(\mathrm{G}), \text{ and } \qquad \psi_2([B]) = b_B \quad \forall\, p \in V(B) \setminus \mathcal{C}(\mathrm{G}).$$

By proposition (4) each equivalence class of $\mathrm{V}/{\sim_B}$ is represented by exactly one of these two forms, this means ψ_2 is a bijection. Now, to examine that ψ_2 preserves adjacency: $[B] \prec_{\sim_B} [c]$ is a covering relation if and only if $c \in \mathrm{V(B)} \cap \mathcal{C}(\mathrm{G})$,this implies that $[B]$ and $[c]$ are adjacent in the Hasse diagram of $\left(\mathrm{V}/{\sim_B}, \preccurlyeq_{\sim_B}\right)$, and because the poset has height exactly one and the covering relation arises strictly between cut-vertex classes and pure vertex classes , which is equivalent to the adjacency condition between c_v and b_B in $\mathcal{BC}(\mathrm{G})$. Therefore, the mapping ψ_2 sends bijectively the Hasse diagram of $\left(\mathrm{V}/{\sim_B}, \preccurlyeq_{\sim_B}\right)$ (after disregarding the edge directions) onto the graph $\mathcal{BC}(\mathrm{G})$, which completes the isomorphism. □

8. **Remark:** An internal bridge block B is represented by block-node b_B in $\mathcal{BC}(\mathrm{G})$, while it has no corresponding equivalence class in $\left(\mathrm{V}/{\sim_B}, \preccurlyeq_{\sim_B}\right)$ since $[B] = \mathrm{V}(B) \setminus \mathcal{C}(\mathrm{G}) = \emptyset$, and therefore do not appear in the Hasse diagram that is a forest not a tree. Consequently, the previous proposition is correspondence bijection iff each block contains at least one pure vertex.

9. **Example:** For the graphs G and H below**,**



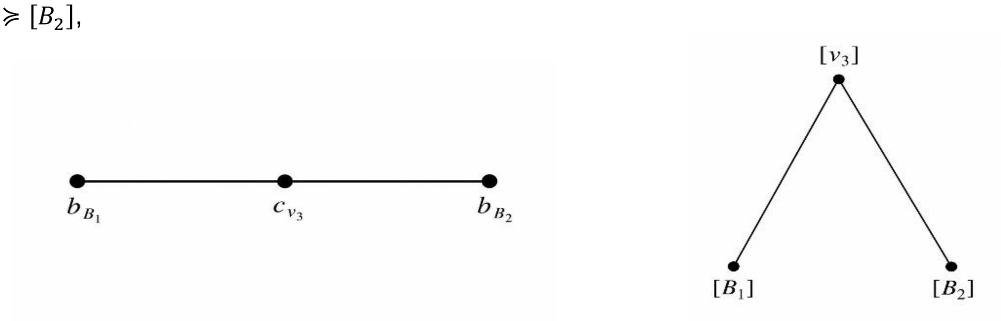
Graph (G) Graph (H)

Figure (1): The graphs G and H of Example (9).

Blocks of G: $B_1 = \{v_1, v_2, v_3\}, B_2 = \{v_3, v_4, v_5\}$, and cut-vertices $\{v_3\}$

$\mathcal{BC}(\mathrm{G})$: a path of length 2: $b_{B_1} - c_{v_3} - b_{B_2}$

$\mathcal{Q}(\mathrm{G})$ has three points: $[B_1] = \{v_1, v_2\}$, $[c] = \{v_3\}$, $[B_2] = \{v_4, v_5\}$ with the covering relations $[B_1] \preccurlyeq [c] \succcurlyeq [B_2]$,



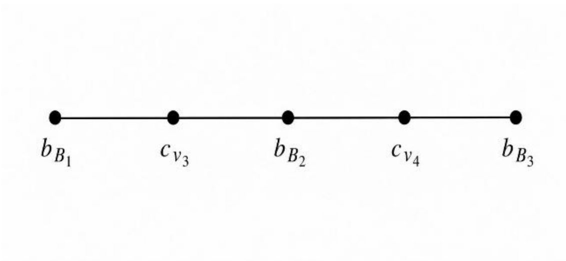
$\mathcal{BC}(\mathrm{G})$ Hasse diagram of $\mathcal{Q}(\mathrm{G})$

Figure (2): $\mathcal{BC}(\mathrm{G})$and the Hasse diagram of $\mathcal{Q}(\mathrm{G})$.

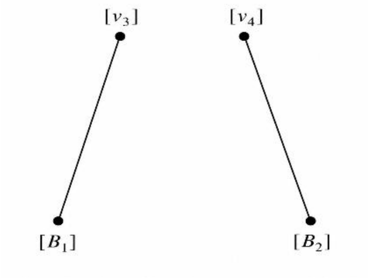
From Figure (2), it is clear that the block-cut tree $\mathcal{BC}(\mathrm{G})$ is isomorphic to the Hasse diagram of $\mathcal{Q}(\mathrm{G})$. while Blocks of H: $B_1 = \{v_1, v_2, v_3\}$, $B_2 = \{v_3, v_4\}$, $B_3 = \{v_4, v_5, v_6\}$, and cut-vertices $\{v_3, v_4\}$

$\mathcal{BC}(\mathrm{H})$: a path of length 4: $b_{B_1} - c_{v_3} - b_{B_2} - c_{v_4} - b_{B_3}$

$\mathcal{Q}(\mathrm{H})$ has four points : $[B_1] = \{v_1, v_2\}$, $[c_1] = \{v_3\}$, $[c_2] = \{v_4\}$, $[B_3] = \{v_5, v_6\}$ with the covering relations $[B_1] \preccurlyeq [c_1]$, $[c_2] \succcurlyeq [B_3]$, note that the bridge block B_2 has no pure vertices so produce no equivalent class .



$\mathcal{BC}(\mathrm{H})$



Hasse diagram of $\mathcal{Q}(\mathrm{H})$

Figure (3): $\mathcal{BC}(\mathrm{H})$and the Hasse diagram of $\mathcal{Q}(\mathrm{H})$.

The block-cut tree $\mathcal{BC}(\mathrm{H})$ has 5 nodes, while Hasse diagram of $\mathcal{Q}(\mathrm{H})$ has 4 nodes which is a forest. Therefore, they are not isomorphic.

Through the Hasse−block-cut correspondence and under the given conditions, Hasse diagram of the quotient poset on $\mathrm{V}/\sim_B$ is represented completely by block-cut tree $\mathcal{BC}(\mathrm{G})$ of the graph G. the topological aspect of this correspondence is demonstrated directly by using the classical theorem of Stong [8], which establishes a natural bijection between partially ordered sets and finite T_0 Alexandroff spaces;

10.**Lemma:** The quotient space $\left(\mathrm{V}/\sim_B, \tau_{\sim_B}(\mathrm{G})\right)$ is Alexandroff T_0 −space.

Proof. Since the block space $(\mathrm{V}, \tau_B(\mathrm{G}))$ is Alexandroff, then its quotient space under the relation $\sim_B$ is again Alexandroff. To verify the T_0 property, assume that $[u] \neq [v]$, then $\mathrm{B}_u \neq \mathrm{B}_v$ and $u \in \mathrm{B}_u \setminus \mathrm{B}_v$, so $[u] \in \pi_{\mathrm{G}}(\mathrm{B}_u)$ and $[u] \notin \pi_{\mathrm{G}}(\mathrm{B}_v)$, therefore, since $\pi_{\mathrm{G}}(\mathrm{B}_u)$ is open in $\tau_{\sim_B}(\mathrm{G})$, the space $\left(\mathrm{V}/\sim_B, \tau_{\sim_B}(\mathrm{G})\right)$ satisfies T_0 axiom. □

The general correspondence between finite Alexandroff T_0 −spaces and posets [2], immediately yields the following proposition;

11.**Proposition:** Let $(\mathrm{V}, \tau_{\mathrm{B}})$ be block space, then the mapping:

$$\psi_1 : \left(\mathrm{V}/\sim_B, \tau_q(\mathrm{G})\right) \longrightarrow \left(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B}\right)$$

is homeomorphism between the quotient space $\left(\mathrm{V}/\sim_B, \tau_q(\mathrm{G})\right)$ and the poset $\left(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B}\right)$ equipped with upper-set topology.

Proof. Let $\psi_1 : \left(\mathrm{V}/\sim_B, \tau_q(\mathrm{G})\right) \longrightarrow \left(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B}\right)$, where ψ_1 maps the quotient space (T_0 Alexandroff) to its partial order via the identity: $\psi_1([v]) = [v]$, that is trivial bijection. It remains to verify ψ_1 and ${\psi_1}^{-1}$ are continuous by showing that a subset $\mathrm{U} \subseteq \mathrm{V}/\sim_B$ is open in $\tau_q(\mathrm{G})$ if and only if it is an upper set of $\preccurlyeq_{\sim_B}$. Suppose that U is open in $\tau_q(\mathrm{G})$, then ${\pi_{\mathrm{G}}}^{-1}(\mathrm{U}) \in \tau_B(\mathrm{G})$, so it is a union of minimal basis elements B_v. If $[u] \in \mathrm{U}$ and $[u] \preccurlyeq_{\sim_B} [v]$, then $v \in \mathrm{B}_u$ and $\mathrm{B}_v \subseteq \mathrm{B}_u \subseteq {\pi_{\mathrm{G}}}^{-1}(\mathrm{U})$, hence $[v] \in \mathrm{U}$. This means that U is an upper set in $\mathrm{V}/\sim_B$. conversely, suppose that U is an upper set of $\preccurlyeq_{\sim_B}$. If $[u] \in \mathrm{U}$ then for each $v \in \mathrm{B}_u$, $[u] \preccurlyeq_{\sim_B} [v]$ implies $[v] \in \mathrm{U}$, therefore ${\pi_{\mathrm{G}}}^{-1}(\mathrm{U})$ is union of B_u for each $[u] \in \mathrm{U}$, thus U is open in $\tau_q(\mathrm{G})$. which proof that ψ_1 is a homeomorphism. □

By combining the proposition (7) and proposition (11), we obtain Hasse-block-cut correspondence;

12.**Theorem:** Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a connected graph without internal bridge blocks, and $\tau_q(\mathrm{G})$ be a quotient topology on $\mathrm{V}/\sim_B$. Then there exists bijective correspondences:

$$\left(\mathrm{V}/\sim_B, \tau_q(\mathrm{G})\right) \xrightarrow{\psi_1} \left(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B}\right) \xrightarrow{\psi_2} \mathcal{BC}(\mathrm{G})$$

Proof. By the propositions (7) and (11), the mappings ψ_1 and ψ_2 are bijections, so the composition $\psi = \psi_2 \circ \psi_1$ is also bijection from the quotient topology $\tau_q(\mathrm{G})$ to the block-cut tree $\mathcal{BC}(\mathrm{G})$ via the quotient poset $\left(\mathrm{V}/\sim_B, \preccurlyeq_{\sim_B}\right)$. □

13.Proposition: Let G and H be two connected graphs . Then**:**

$$\mathcal{Q}(\mathrm{G}) \cong \mathcal{Q}(\mathrm{H}) \iff \mathcal{BC}(\mathrm{G}) \cong \mathcal{BC}(\mathrm{H})$$

Proof. From theorem (12), there exist bijections;

$$\psi_{\mathrm{G}} : \mathcal{Q}(\mathrm{G}) \longrightarrow \mathcal{BC}(\mathrm{G}) \quad \text{and} \quad \psi_{\mathrm{H}} : \mathcal{Q}(\mathrm{H}) \longrightarrow \mathcal{BC}(\mathrm{H})$$

Suppose that $\mathcal{Q}(\mathrm{G}) \cong \mathcal{Q}(\mathrm{H})$, and $\phi: \mathcal{Q}(\mathrm{G}) \longrightarrow \mathcal{Q}(\mathrm{H})$ is an isomorphism, so the composition:

$$\mathcal{BC}(\mathrm{G}) \xrightarrow{\psi_{\mathrm{G}}^{-1}} \mathcal{Q}(\mathrm{G}) \xrightarrow{\phi} \mathcal{Q}(\mathrm{H}) \xrightarrow{\psi_{\mathrm{H}}} \mathcal{BC}(\mathrm{H})$$

is an isomorphism, this means that $\mathcal{BC}(\mathrm{G}) \cong \mathcal{BC}(\mathrm{H})$. The converse implication is proved by the same way. □

14.Corollary: For each connected graph G without internal bridge blocks. The quotient space $\mathcal{Q}(\mathrm{G})$ is a complete invariant of the block-cut tree $\mathcal{BC}(\mathrm{G})$.

Proof. From theorem (12), there is an isomorphism between the Hasse diagram of the specialization poset of $\mathcal{Q}(\mathrm{G})$ and block-cut tree $\mathcal{BC}(\mathrm{G})$, but $\mathcal{Q}(\mathrm{G})$ is Alexandroff T_0 –space, so it determines completely by Hasse diagram of its specialization poset. Consequently, $\mathcal{Q}(\mathrm{G})$ is completely determined by $\mathcal{BC}(\mathrm{G})$. conversely, $\mathcal{BC}(\mathrm{G})$ can be recovered strictly from the Hasse diagram of $\mathcal{Q}(\mathrm{G})$, so both structures $\mathcal{Q}(\mathrm{G})$ and $\mathcal{BC}(\mathrm{G})$ has equivalent information. □

15.**Definition:** Let $(\mathcal{P}, \preccurlyeq)$ be finite poset, and $x \in \mathcal{P}$. Define the Hasse neighbor $\mathcal{N}_{\mathrm{H}}(x)$ of x by:

$$\mathcal{N}_{\mathrm{H}}(x) = \{y \in \mathcal{P} : y \prec x \quad \text{or} \quad x \prec y\}$$

Such that there is no $z \in \mathcal{P}$ satisfies $x \prec z \prec y$. The points in Hasse diagram of $\mathcal{P}$ is equal the elements of $\mathcal{P}$, and an edge between x and y if they are Hasse neighbors.

The following proposition shows that the degree of any node in the block-cut tree is determined directly from corresponding quotient space;

16.Theorem: Let G be a connected graph without internal bridge blocks, and $\mathcal{Q}(\mathrm{G})$ be its quotient topology. If $[x] \in \mathcal{Q}(\mathrm{G})$, then**:**

$$d_{\mathcal{BC}(\mathrm{G})}(x) = \begin{cases} |\{[B]:\ c \in \mathrm{V}(B)\}| & ; x = c \\ \\ |\{[c]:\ c \in \mathrm{V}(B)\}| & ;\ x = b_B \end{cases}$$

Proof. By proposition (7), the Hasse diagram of quotient space $\mathcal{Q}(\mathrm{G})$ is isomorphic to the block-cut tree $\mathcal{BC}(\mathrm{G})$, and each point in Hasse diagram has number of its adjacent elements equals (under covering relation) to its Hasse neighbors, which is exactly the degree of its corresponding node in $\mathcal{BC}(\mathrm{G})$. Now, since the quotient poset has height 1, then each point of Hasse diagram is either block class or cut-vertex class;

- **Case 1:** if $[x] = [c]$, since $c \in \mathcal{C}(\mathrm{G})$, then it is maximal and so $\mathcal{N}_{\mathrm{H}}([c]) = \{[y] \in \mathcal{Q}(\mathrm{G}) : [y] \prec [c]\} = \{[B] : [B] \prec [c]\}$, therefore $d_{\mathrm{Hasse}}([c]) = |\mathcal{N}_{\mathrm{H}}([c])| = |\{[B]:\ c \in \mathrm{V}(B)\}|$.
- **Case 2:** if $[x] = [B]$, then it is minimal and so $\mathcal{N}_{\mathrm{H}}([B]) = \{[y] \in \mathcal{Q}(\mathrm{G}) : [B] \prec [y]\} = \{[c] : [B] \prec [c]\}$, therefore $d_{\mathrm{Hasse}}([B]) = |\mathcal{N}_{\mathrm{H}}([B])| = |\{[c]:\ c \in \mathrm{V}(B)\}|$.

In both cases, and from $d_{\mathcal{BC}(\mathrm{G})}(x) = d_{\mathrm{Hasse}}([x])$, the desired formula holds. □

17.**Example:** In the graph G of Example (9), note that $\mathcal{N}_{\mathrm{H}}([v_3]) = \{[B_1], [B_2]\}$, and $\mathcal{N}_{\mathrm{H}}([B_1]) = \mathcal{N}_{\mathrm{H}}([B_2]) = \{[v_3]\}$, so $d_{\mathcal{BC}(\mathrm{G})}(c_{v_3}) = 2$ and $d_{\mathcal{BC}(\mathrm{G})}(b_{B_1}) = d_{\mathcal{BC}(\mathrm{G})}(b_{B_2}) = 1$.

18.**Proposition:** Let G and H be connected graphs, and $\phi : \mathrm{G} \to \mathrm{H}$ be a graph isomorphism, then the induced map $\tilde{\phi}: \mathcal{Q}(\mathrm{G}) \to \mathcal{Q}(\mathrm{H})$ that defined by $\tilde{\phi}([v]) = [\phi(v)]$ is a homeomorphism.

Proof. Suppose that ϕ is a graph isomorphism between the graphs G and H , then ϕ preserves the cut-vertices and blocks and adjacency structure. So, for each vertex v, we have $\phi(\mathrm{B}_v) = \mathrm{B}_{\phi(v)}$ and:

$$[u] \preccurlyeq [v] \iff \mathrm{B}_u \supseteq \mathrm{B}_v \iff \phi(\mathrm{B}_u) \supseteq \phi(\mathrm{B}_v) \iff \mathrm{B}_{\phi(u)} \supseteq \mathrm{B}_{\phi(v)}$$

$$\iff [\phi(u)] \supseteq [\phi(v)] \iff \tilde{\phi}([u]) \supseteq \tilde{\phi}([v])$$

Hence $\tilde{\phi}$ is an isomorphism between the specialization posets of $\mathcal{Q}(\mathrm{G})$ and $\mathcal{Q}(\mathrm{H})$, but each finite Alexandroff T_0 –space is exactly determined by their specialization orders; thus, every order isomorphism is a homeomorphism, and $\tilde{\phi}$ is homeomorphism. □

The following theorem is specifically derived from the previous proposition;

19.Proposition: Let G be connected graph. Every graph automorphism of G induces homeomorphism of the quotient space $\mathcal{Q}(\mathrm{G})$.

Proof. By applying the proposition (18) with $\mathrm{H} = \mathrm{G}$, we obtain the homeomorphism $\tilde{\phi} : \mathcal{Q}(\mathrm{G}) \to \mathcal{Q}(\mathrm{G})$. □

Conclusion:

We showed that the quotient space $\mathcal{Q}(\mathrm{G})$ is a finite Alexandroff T_0-space whose Hasse diagram is isomorphic to the block-cut tree, provided G has no internal bridge blocks. This correspondence is strong enough that either structure can be recovered from the other without requiring block size information, unlike in the classification of [6]. As a direct consequence, we obtain a uniform degree formula and prove that every graph isomorphism induces a homeomorphism.

When G contains internal bridge blocks, the correspondence loses bijectivity, since such blocks carry no corresponding equivalence class in $\mathcal{Q}(\mathrm{G})$, causing the Hasse diagram to become a forest. Future work can address this limitation by refining the equivalence relation or by using a filtered quotient. Key extensions include generalizing to study directed graphs via strongly connected components, and applying finite topological tools, such as homotopy and cohomology, to $\mathcal{Q}(\mathrm{G})$.

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